**Colpitts Oscillator**

- Basic positive feedback oscillator
- The Colpitts LC Oscillator circuit
- Open-loop analysis
- Closed-loop analysis
- Root locus
- Stability limit
- Colpitts design
**Basic Positive Feedback Oscillator**

**Closed-loop oscillator**

\[ V_i = 0 \rightarrow \begin{array}{c} \text{Pos. fdbk} \end{array} \rightarrow \frac{V_i}{V_f} \rightarrow A(s) \rightarrow V_o \]

\[ V_o = A(s)(V_i + V_f) = A(s)(0 + V_o) \Rightarrow \]

\[ V_o(1 - A(s)) = 0 \]

Since: \( V_o \neq 0 \Rightarrow 1 - A(s) = 0 \Rightarrow A(s) = 1 \Rightarrow D(s) - K N(s) = 0 \]

**Condition for oscillation at** \( s = j \omega_0 : \]

\[ A(s) = 1 e^{j \pm 2k \pi} \text{ Barkhausen criterion} \]

**Open-loop: determine loop-gain**

\[ V_i \rightarrow \begin{array}{c} \text{Pos. fdbk} \end{array} \rightarrow \frac{V_i}{V_f} \rightarrow A(s) \rightarrow V_o \]

\[ \frac{V_f}{V_i} = \frac{V_o}{V_i} = A(s) = \frac{K N(s)}{D(s)} \]
Colpitts Oscillator Basic Schematic

Assumptions:
1. $r_\pi$ large (compared to $1/\omega C_2$).
2. $C_\mu$ negligible (compared to $C_1$, $C_2$)
3. $C_\pi$ part of $C_2$ (in closed loop)
4. $R$ represents total resistance in collector circuit, i.e. $R||r_o \approx R$
Loop-Gain Analysis

Node equation at $v_{cg}$:

$$\left(g_m - sC_\mu\right)V_{\pi} + \frac{V_{cg}}{R} + s\left(C_1 + C_\mu\right)V_{cg} + \frac{V_{cg} - V_f}{sL} = 0$$

at $v_f$:

$$\frac{V_f - V_{cg}}{sL} + sC_2 V_f = 0$$

Note that:

$$V_f = V_f(s)$$
$$V_{cg} = V_{cg}(s)$$
$$V_{\pi} = V_{\pi}(s)$$

$C_\mu \ll C_1$
Open Loop Analysis - cont.

Rearranging the two equations:

\[
(sC_1 + \frac{1}{sL} + \frac{1}{R}) V_{cg} - \frac{1}{sL} V_f = (sC_\mu - g_m) V_\pi \\
- \frac{1}{sL} V_{cg} + (sC_2 + \frac{1}{sL}) V_f = 0
\]

Further rearrangement:

\[
\left(\frac{s^2 LC_1 + 1}{sL} + \frac{1}{R}\right) V_{cg} - \frac{1}{sL} V_f = (sC_\mu - g_m) V_\pi \\
- \frac{1}{sL} V_{cg} + \left(\frac{s^2 LC_2 + 1}{sL}\right) V_f = 0
\]

from previous slide

\[
(g_m - sC_\mu) V_\pi + \frac{V_{cg}}{R} + s(C_1 + C_\mu) V_{cg} + \frac{V_{cg} - V_f}{sL} = 0 \\
\frac{V_f - V_{cg}}{sL} + sC_2 V_f = 0
\]
Open Loop Analysis - cont.

Prepare to add the two equations:

\[
\frac{1}{sL} \left( \frac{s^2 LC_1 + 1}{sL} + \frac{1}{R} \right) V_{cg} - \left( \frac{1}{sL} \right)^2 V_f = \frac{s C_\mu - g_m}{sL} V_f
\]

\[-\frac{1}{sL} \left( \frac{s^2 LC_1 + 1}{sL} + \frac{1}{R} \right) V_{cg} + \left( \frac{s^2 LC_1 + 1}{sL} + \frac{1}{R} \right) \left( \frac{s^2 LC_2 + 1}{sL} \right) V_f = 0
\]

Adding \( V_{cg} \) terms cancel:

\[
\left( -\left( \frac{1}{sL} \right)^2 + \left( \frac{s^2 LC_1 + 1}{sL} + \frac{1}{R} \right) \left( \frac{s^2 LC_2 + 1}{sL} \right) \right) V_f = \frac{s C_\mu - g_m}{sL} V_f
\]

from previous slide

\[
\left( \frac{s^2 LC_1 + 1}{sL} + \frac{1}{R} \right) V_{cg} - \frac{1}{sL} V_f = (s C_\mu - g_m) V_\pi \tag{1}
\]

\[-\frac{1}{sL} V_{cg} + \left( \frac{s^2 LC_2 + 1}{sL} \right) V_f = 0 \tag{2}
\]

\[\text{Eq (1) } \ast \frac{1}{sL} \]

\[\text{Eq (2) } \ast \left( \frac{s^2 LC_1 + 1}{sL} + \frac{1}{R} \right) \]
Open Loop Analysis – cont.

From previous slide

\[
\left( -\left( \frac{1}{sL} \right)^2 + \left( \frac{s^2 L C_1 + 1}{sL} + \frac{1}{R} \right) \left( \frac{s^2 L C_2 + 1}{sL} \right) \right) V_f = \frac{C_\mu - g_m}{sL} V_\pi
\]

Multiply by \((sL)^2\):

\[
\left( -1 + \left( s^2 L C_1 + 1 + \frac{sL}{R} \right) \left( s^2 L C_2 + 1 \right) \right) V_f = \left( s C_\mu - g_m \right) sL V_\pi
\]

Expand and collect terms according to \(s^n\):

\[
\left( -1 + s^4 C_1 C_2 L^2 + s^2 \left( L C_1 + L C_2 \right) + s^3 \frac{L^2 C_2}{R} + s \frac{L}{R} + 1 \right) V_f = \left( s C_\mu - g_m \right) sL V_\pi
\]
Open Loop Analysis - cont.

From previous slide

\[
\left( -1 + s^4 C_1 C_2 L^2 + s^2 (L C_1 + L C_2) + \frac{s^3 L^2 C_2}{R} + \frac{s L}{R} + 1 \right) V_f = \left( s C_\mu - g_m \right) s L V_\pi
\]

Canceling \((-1)\) by \(1\) and dividing by \(sL\):

\[
\left( s^3 C_1 C_2 L + s (C_1 + C_2) + \frac{s^2 L C_2}{R} + \frac{1}{R} \right) V_f = \left( s C_\mu - g_m \right) V_\pi
\]

Multiply by \(R\):

\[
\left( s^3 R C_1 C_2 L + s R (C_1 + C_2) + s^2 L C_2 + 1 \right) V_f = \left( s C_\mu - g_m \right) R V_\pi
\]
Open Loop Analysis - cont.

From previous slide

\[
\left(s^3 \frac{R C_1 C_2 L + s R (C_1 + C_2) + s^2 L C_2 + 1}{R C_1 C_2 L}ight) V_f = \left(s C_\mu - g_m\right) R V_\pi
\]

Normalize (divide by \(R C_1 C_2 L\)) and factor out \(C_\mu\):

\[
\left(s^3 + s \frac{C_1 + C_2}{C_1 C_2 L} + s^2 \frac{1}{RC_1} + \frac{1}{RC_1 C_2 L}\right) V_f = \frac{(s - \frac{g_m}{C_\mu}) R C_\mu}{RC_1 C_2 L} V_\pi
\]

The loop-gain transfer function:

\[
\frac{V_f}{V_\pi} = A(s) = \frac{R C_\mu}{R C_1 C_2 L} \frac{(s - \frac{g_m}{C_\mu})}{s^3 + s^2 \frac{1}{RC_1} + s \frac{C_1 + C_2}{C_1 C_2 L} + \frac{1}{RC_1 C_2 L}} = \frac{K N(s)}{D(s)}
\]

2008 Kenneth R. Laker (based on P. V. Lopresti 2006) updated 03Nov08 KRL
Closed Loop Analysis - cont.

The closed loop equation (note the positive feedback):

\[
T(s) = \frac{A(s)}{1 - A(s)} = \frac{K N(s)}{D(s) - K N(s)}
\]

where:

\[
A(s) = \frac{K N(s)}{D(s)} \quad K = \frac{R C^\mu}{R C_1 C_2 L} \quad N(s) = s - \frac{g_m}{C^\mu} \approx -\frac{g_m}{C^\mu}
\]

\[
D(s) = s^3 + s^2 \frac{1}{R C_1} + s \frac{(C_1 + C_2)}{C_1 C_2 L} + \frac{1}{R C_1 C_2 L}
\]

then:

\[
D(s) - K N(s) = D(s) - \frac{-g_m R}{R C_1 C_2 L} = s^3 + s^2 \frac{1}{R C_1} + s \frac{(C_1 + C_2)}{C_1 C_2 L} + \frac{1 + g_m R}{R C_1 C_2 L}
\]
Closed Loop Analysis - cont.

We know that the open loop system \( A(s) \) is stable. It has poles in the left-half \( s \)-plane, since it is a passive RLC circuit. We also know that it has 3 stable poles. One is negative-real, the other 2 can be negative-real or LHP complex conjugates.

\[
D(s) = s^3 + s^2 \frac{1}{RC_1} + s \left( \frac{C_1 + C_2}{C_1 C_2 L} \right) + \frac{1}{RC_1 C_2 L}
\]

So, let's do a rough sketch of the root locus for a feedback system with a 3 stable pole \( A(s) \).
Root Locus Characteristic

The loop will become unstable for any value of \( K > K_0 \).

Rather than sketch the root locus in more exacting detail – it has served its purpose by verifying that oscillation is possible.

Let's solve for the required \( K_0 \).
Stability Limit Calculation

If the closed-loop system is at the stability limit point:

\[ D(s) = (s + a)(s^2 + \omega_x^2) \]

Multiplying terms:

\[ D(s) = s^3 + a s^2 + \omega_x^2 s + a \omega_x^2 \Rightarrow D(j \omega) = (a \omega_x^2 - a \omega_x^2) = j(\omega_x \omega - \omega^3) \]

Match term by term with:

\[ D(s) + KN(s) = s^3 + s^2 \left( \frac{1}{RC_1} \right) + s \left( \frac{C_1 + C_2}{C_1 C_2 L} \right) + \frac{1 + g_m R}{RC_1 C_2 L} \]

\[ a = \frac{1}{RC_1} \quad \omega_x^2 = \frac{1}{\left( \frac{C_1 C_2}{C_1 + C_2} \right) L} \quad a \omega_x^2 = \frac{1 + g_m R}{RC_1 C_2 L} \]

to oscillate
Stability Limit

\[
a = \frac{1}{RC_1} \\
\omega_x^2 = \frac{1}{\left(\frac{C_1C_2}{C_1 + C_2}\right)C_1C_2L} = \frac{C_1 + C_2}{C_1C_2} \\
a \omega_x^2 = \frac{1 + g_m R}{RC_1C_2L}
\]

To find the “gain” requirement for oscillation, equate:

\[
\frac{1}{RC_1} \frac{C_1 + C_2}{C_1C_2L} = \frac{C_1 + C_2}{RC_1C_2L} = \frac{1 + g_m R}{RC_1C_2L} \Rightarrow 1 + g_m R = \frac{C_1 + C_2}{C_1}
\]

\[
g_m R = \frac{C_1 + C_2 - C_1}{C_1} = \frac{C_2}{C_1}
\]
**Oscillator Design Summary**

The oscillation frequency:

\[ \omega_x = \sqrt{\frac{C_1 + C_2}{C_1 C_2 L}} \]

The required feedback gain \( C_2 / C_1 \):

\[ g_m R = \frac{C_2}{C_1} \]

To insure the start-up of oscillation:

\[ g_m R > \frac{C_2}{C_1} \]

Comments:

1. Unfortunately we don't control “\( R \)”.
2. We can fix \( g_m \) and adjust \( C_2 / C_1 \) and adjust “\( L \)” to keep \( \omega_x \) constant.
3. We can adjust \( g_m \) through the bias current \( I_C \) and set \( C_2 \) and \( C_1 \) at convenient values, say \( C_2 = C_1 = C \). We can now choose \( L \).
Practical Colpitts Oscillator Circuit

RFC = RF choke ->
large reactance at $\omega_0$
low resistance at dc