Chapter 4: Analysis of fluid motion: Integral Formulation

4 Preliminaries

4.1 Lagrangian and Eulerian descriptions

**Kinematics:** is simply the description of fluid flow in terms of displacements, velocities, and accelerations without regard to the various forces which cause these. Clearly, streamline, pathline and streakline descriptions are all part of kinematic descriptions. Every particle of fluid in a flow has an instantaneous value of velocity and density as well as other properties. As the particle moves about in the fluid, its velocity as well as its density, and so on, will change from point to point and from time to time. The detailed history of each individual mass particle as it moves about is called the **Lagrangian description.** Lagrangian description is usually difficult to provide if there are many particles. However, rather than follow the history of each individual particle, it is found more convenient to describe the flow by giving the velocity components, pressure, density, and so on, of the flow at every point of space as a function of time. This is called the **Eulerian description.** In practice, we specify any desired location in the flow and then state the magnitude of the flow variables of interest. Now, the variables are described in a suitable coordinate system. For example, in the cartesian coordinate system, velocity \( \mathbf{V} = i u(x, y, z, t) + j v(x, y, z, t) + k w(x, y, z, t) \) where \( u(x, y, z, t), v(x, y, z, t), w(x, y, z, t) \) are the components of the fluid velocity in the \( i, j, k \) directions.

- Recall that the acceleration of a fluid particle is ascribable to two reasons. One reason is due to the fluid particle being “convected” from a given location to another location in the flow, the second location being of higher or lower velocity. This is called “Convective acceleration.” Some people call this “Adective” acceleration, and keep the description convective acceleration to describe similar situations but with heat transport present in addition. The second reason is due to the “unsteadiness” of the flow, that is, due to the change in the “local” velocity of the fluid particle as a function of time. This is called “Local acceleration.”

- The velocity and acceleration of a fluid particle in the Lagrangian description are simply the partial time derivatives. Thus,

\[
\begin{align*}
    u &= \frac{\partial x}{\partial t}, \\
    v &= \frac{\partial y}{\partial t}, \\
    w &= \frac{\partial z}{\partial t} \\
    a_x &= \frac{\partial u}{\partial t}, \\
    a_y &= \frac{\partial v}{\partial t}, \\
    a_z &= \frac{\partial w}{\partial t} \quad \text{(226)}
\end{align*}
\]

- In the Eulerian description, however, the partial derivative \( \frac{\partial}{\partial t} \) gives only the local rate of change at \((x, y, z)\), and is not the total rate of change “observed” by a fluid
particle. Here, we have to evaluate the rate of change of velocity at each point in the flow following a particle of fixed identity. In other words, we need to express a Lagrangian concept in Eulerian terms.

Let \( V(x, y, z, t) \) and \( V + dV(x + dx, y + dy, z + dz, t + dt) \) be the velocities of a given particle in a flow field at \((x, y, z, t)\) and at \((x + dx, y + dy, z + dz, t + dt)\), respectively. The quantities, \((x, y, z, t)\) and \((x + dx, y + dy, z + dz, t + dt)\) could be along its particle path. Then, from calculus,

\[
\frac{dV}{dt} = \frac{\partial V}{\partial t} + \frac{\partial V}{\partial x} dx + \frac{\partial V}{\partial y} dy + \frac{\partial V}{\partial z} dz. \tag{227}
\]

\[
\frac{dV}{dt} = \frac{\partial V}{\partial t} + \frac{\partial V}{\partial x} dx + \frac{\partial V}{\partial y} dy + \frac{\partial V}{\partial z} dz. \tag{228}
\]

The expression 228 is called the total derivative of velocity of the flow and is the acceleration vector, \(a\), of the flow. In 228, \(\frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt}\) may be interpreted as the components of the velocity of an "observer" moving in the fluid who is observing the velocity changes of the fluid particle as it is advected from location \((x, y, z)\) to location \((x + dx, y + dy, z + dz)\) in the flow. \(\frac{\partial V}{\partial x} dx + \frac{\partial V}{\partial y} dy + \frac{\partial V}{\partial z} dz\) is the advective acceleration. The term \(\frac{\partial V}{\partial t}\) is the local acceleration of the particle.

The total derivative is too general and will account for the acceleration whether the observer is moving along with the fluid or against the flow or in any arbitrary manner. Such generality is seldom necessary and in fluid mechanics, we prefer to study the particle motion in terms of an "observer" who is moving along with the fluid flow and with a velocity same as that of the flow. In this case, \(\frac{dx}{dt} = u, \frac{dy}{dt} = v, \frac{dz}{dt} = w\). Then we denote the acceleration vector by

\[
\frac{DV}{Dt} = a = \frac{\partial V}{\partial t} + \frac{\partial V}{\partial x} u + \frac{\partial V}{\partial y} v + \frac{\partial V}{\partial z} w. \tag{229}
\]

The representation 229 is called the Material derivative or Substantial derivative following the motion. Now recall,

\[
\nabla = i \frac{\partial}{\partial x} + j \frac{\partial}{\partial y} + k \frac{\partial}{\partial z}. \tag{230}
\]

Therefore, \(\mathbf{V} \cdot \nabla = (i \mathbf{u} + j \mathbf{v} + k \mathbf{w}) \cdot \left( i \frac{\partial}{\partial x} + j \frac{\partial}{\partial y} + k \frac{\partial}{\partial z} \right) \tag{231}\)

Thus, \(\mathbf{V} \cdot \nabla = \left( u \frac{\partial}{\partial x} + v \frac{\partial}{\partial y} + w \frac{\partial}{\partial z} \right) \). \tag{232}
Therefore, from 229,
\[
\frac{DV}{Dt} = a = \frac{\partial V}{\partial t} + (V \cdot \nabla) V
\]  
(233)

In component form,
\[
a = \left( i \frac{\partial u}{\partial t} + j \frac{\partial v}{\partial t} + k \frac{\partial w}{\partial t} \right) + u \left( i \frac{\partial u}{\partial x} + j \frac{\partial v}{\partial x} + k \frac{\partial w}{\partial x} \right) + v \left( i \frac{\partial u}{\partial y} + j \frac{\partial v}{\partial y} + k \frac{\partial w}{\partial y} \right) + w \left( i \frac{\partial u}{\partial z} + j \frac{\partial v}{\partial z} + k \frac{\partial w}{\partial z} \right)
\]
(234)

This can be rearranged:
\[
a = i \left[ \frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} + w \frac{\partial u}{\partial z} \right]
\]  
(235)

\[
+ j \left[ \frac{\partial v}{\partial t} + u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} + w \frac{\partial v}{\partial z} \right]
\]  
(236)

\[
+ k \left[ \frac{\partial w}{\partial t} + u \frac{\partial w}{\partial x} + v \frac{\partial w}{\partial y} + w \frac{\partial w}{\partial z} \right]
\]  
(237)

### 4.2 System and Control volume concepts

**System:** Identified quantity of matter. A system may change shape, position, and thermal condition but must always involve the same matter. Lagrangian concept. There is no mass transfer across the system.

**Control volume:** is an arbitrary volume in space through which the fluid flows. Eulerrian concept. The amount and identity of matter in the cv may change with time. The boundary of this volume is called the control surface (cs). The cv may be a moving volume and may deform.

The cv study is preferred in most situations of interest.

We can relate the system approach to the cv approach for certain fluid and flow properties through the use of the definitions of “Extensive”\((B)\) and “Intensive”\((b)\) properties. Properties of a fluid which depend on the mass present (amount of the substance) are called Extensive properties (mass, volume, linear momentum, angular momentum, energy, entropy etc.). Those that do not depend on the amount of substance present are called Intensive properties (specific volume, velocity, specific energy, specific entropy, pressure, temperature etc.)
4.3 Basic laws for analyzing a fluid flow

(1) Conservation of mass (2) Conservation of momentum (Linear, Angular) (3) Conservation of energy (First law of Thermodynamics) (4) Entropy principle (Second law of Thermodynamics). In addition to these, we may need the equation of state, Newton’s law of viscosity, Hooke’s law for elastic materials etc. These form part of “Constitutive” laws. We have to satisfy all of these laws.

4.4 Integral and Differential Analyses of fluid flow using cvd

- There are two ways of analyzing fluid motion using cvd: Integral analysis and Differential analysis.

- In many fluid flow problems of engineering interest, it may be sufficient to know only the average values of flow quantities or only components of resultant forces acting on the system or the cv. It may be unnecessary to get involved with the details of the flow. (Example: The design of a lawn sprinkler). In such cases, integral analysis is useful. Here, we consider basic laws for finite systems, and using a theorem called the Reynolds transport theorem that enables us to change from a system approach to a cv approach, we will formulate the basic laws for finite cvd. Then we solve the integral formulation for resultant force components, average velocities etc. Typically, conditions on the surfaces of the cvd will matter the most in these type of problems.

- Where we need detailed information regarding the flow, we have to employ the Differential analysis. Here, we state the conservation laws in a differential form by developing appropriate differential equations valid at a point in the flow. These developments are based on mass, momentum and energy balances on elemental cvd. The differential equations have to be integrated using the applicable initial and boundary conditions.

4.5 Integral Analysis

4.5.1 Basic Laws for a system

First let us write down the basic laws for a system.

- Conservation of Mass of a system
For system, by definition, the mass \( M = \text{constant} \). Note that \( M \) is an extensive property. Now, with \( V_{sys} \) denoting volume of system,

\[
\frac{dM}{dt}\bigg|_{sys} = 0, \quad \text{where,} \quad M|_{sys} = \int_{sys} dm = \int_{V_{sys}} \rho \, dV_{sys} \quad (238)
\]

- Conservation of Linear momentum of a system

From Newton's law, with \( P \) denoting the linear momentum of a system, the time rate of change of linear momentum of the system is equal to the sum of all the external forces, \( F \), acting on the system. Note that \( P \) is an extensive property. Newton's law is:

\[
F = \frac{dP}{dt}\bigg|_{sys}, \quad \text{where,} \quad P|_{sys} = \int_{sys} V \, dm = \int_{V_{sys}} V \rho \, dV_{sys} \quad (239)
\]

- The Angular momentum principle

The rate of change of angular momentum, \( H \), for a system is equal to the sum of all torques, \( T \) acting on the system. Note that \( H \) is an extensive property. The angular momentum principle may be expressed as:

\[
T = \frac{dH}{dt}\bigg|_{sys}, \quad \text{where,} \quad H|_{sys} = \int_{sys} \mathbf{r} \times \mathbf{V} \, dm = \int_{V_{sys}} \mathbf{r} \times \mathbf{V} \rho \, dV_{sys} \quad (240)
\]

We note that torque \( T \) can be produced by surface and body forces and also by shafts that cross the system boundary,

\[
T = \mathbf{r} \times \mathbf{F}_{surface} + \int_{sys} \mathbf{r} \times \mathbf{g} \, dm + T_{shaft} \quad (241)
\]

- The First Law of Thermodynamics

Law of conservation of energy, \( E \), for a system. Note that \( E \) is an extensive property. For \( Q \), heat added, and \( W \), external work done by the system, the first law may be written as:

\[
\dot{Q} = \frac{dE}{dt}\bigg|_{sys} + \dot{W}, \quad \text{where,} \quad E|_{sys} = \int_{sys} e \, dm = \int_{V_{sys}} e \rho \, dV_{sys}, \quad \text{where,} \quad e = \dot{u} + \frac{V^2}{2} + g \, z \quad (242)
\]

- The Entropy principle

Let \( S \) denoting the entropy function, \( \delta Q \), the heat added, and, \( T \), the temperature at which the transfer occurs. Note that \( S \) is an extensive property. From second law of thermodynamics,

\[
dS \geq \frac{\delta Q}{T} = \left( \frac{dS}{dt}\right)_{sys} \geq \frac{\dot{Q}}{T}, \quad \text{where,} \quad S|_{sys} = \int_{sys} s \, dm = \int_{V_{sys}} s \rho \, dV_{sys} \quad (243)
\]
The equations 238, 239, 240, 242, 243, are the basic laws for a system in terms of extensive properties, \( M, \mathbf{P}, \mathbf{H}, E, \) and \( S. \) We shall denote the general extensive property by \( B. \) The corresponding intensive properties are: 1, \( \mathbf{V}, (\mathbf{r} \times \mathbf{V}), e, \) and \( s. \) We shall denote the general intensive property by \( b. \)

We now have to change over from a system viewpoint to a control volume viewpoint. So we have to express, \( \frac{dM}{dt} \big|_{\text{sys}} \), \( \frac{d\mathbf{P}}{dt} \big|_{\text{sys}} \), \( \frac{d\mathbf{H}}{dt} \big|_{\text{sys}} \), \( \frac{dE}{dt} \big|_{\text{sys}} \), \( \left( \frac{ds}{dt} \right) \big|_{\text{sys}} \), in terms of changes that can be associated with cv's. An important theorem called the Reynolds Transport Theorem lets us do that.

### 4.5.2 Reynolds Transport Theorem

\[
\frac{dB}{dt} \bigg|_{\text{sys}} = \frac{\partial}{\partial t} \int_{cv} b \rho \, dV_{cv} + \int_{cs} b \rho \mathbf{V} \cdot \mathbf{n} \, dA_{cs} \tag{244}
\]

We note that in here, the term \( \frac{dM}{dt} \big|_{\text{sys}} \) is the total rate of change of any chosen extensive property, \( B, \) of the system. The expression \( \int_{cv} b \rho \, dV_{cv} \) on the right hand side is the total amount of \( B \) contained within the cv. Therefore, the term \( \frac{\partial}{\partial t} \int_{cv} b \rho \, dV_{cv} \) is the time rate of change of \( B \) within the cv. The term \( b \rho \mathbf{V} \cdot \mathbf{n} \, dA_{cs} \) is the rate of mass flux through the area element \( \mathbf{n} \, dA_{cs} \) per unit time. We can think of \( \mathbf{n} \, dA_{cs} \) as area vector \( dA_{cs}. \) We note that the dot product is a scalar product; the sign of \( \rho \mathbf{V} \cdot \mathbf{n} \) depends on the direction of the velocity vector, \( \mathbf{V}, \) relative to the area vector \( \mathbf{n} \, dA_{cs} = dA_{cs}. \) Next, \( b \rho \mathbf{V} \cdot \mathbf{n} \, dA_{cs} \) is the rate of flux of \( B \) through the area \( dA_{cs}. \) Finally, \( \int_{cs} b \rho \mathbf{V} \cdot \mathbf{n} \, dA_{cs} \) is the net rate of flux of \( B \) out through the control surface \( cs. \)

- The total rate of change of \( B \) of the system is equal to the time rate of change of \( B \) within the cv plus the net rate of flux of \( B \) out through the control surface \( cs. \)

### 4.5.3 Application of Reynolds Transport Theorem and Basic laws for use with cv's

- Conservation of Mass

Here, \( B = M \) and \( b = 1. \) Therefore, from, 244,

\[
\frac{dM}{dt} \bigg|_{\text{sys}} = \frac{\partial}{\partial t} \int_{cv} 1 \, \rho \, dV_{cv} + \int_{cs} 1 \rho \mathbf{V} \cdot \mathbf{n} \, dA_{cs} \tag{245}
\]

But, \( \frac{dM}{dt} \big|_{\text{sys}} = 0. \) Therefore, for fluid flow through a cv,

\[
0 = \frac{\partial}{\partial t} \int_{cv} \rho \, dV_{cv} + \int_{cs} \rho \mathbf{V} \cdot \mathbf{n} \, dA_{cs} \tag{246}
\]
On the right hand side, the first term represents the rate of change of mass within the cv; the second term represents the net rate of mass flux out through the cs. Conservation of mass, therefore, requires that the sum of the rate of change of mass within the cv and the net rate of mass outflow through the control surface be zero. The equation 246 would apply to a compressible, unsteady flow.

For an incompressible flow,

\[
0 = \rho \frac{\partial}{\partial t} \int_{cv} dV_{cs} + \rho \int_{cs} \mathbf{V} \cdot \mathbf{n} \, dA_{cs} \quad (247)
\]

\[
0 = \frac{\partial V_{cs}}{\partial t} + \int_{cs} \mathbf{V} \cdot \mathbf{n} \, dA_{cs} \quad (248)
\]

Therefore, for a non-deformable cv, that is, a cv of fixed size and shape, the conservation of mass for an incompressible flow through the cv, steady or unsteady does not matter, becomes:

\[
0 = \int_{cs} \mathbf{V} \cdot \mathbf{n} \, dA_{cs} = \int_{cs} \mathbf{V} \cdot dA_{cs} \quad (249)
\]

The quantity \( \int_{cs} \mathbf{V} \cdot dA_{cs} \) is the volume rate of flow, \( Q \), and for an incompressible flow, the volume flow rate into a fixed cv must be equal to the volume flow rate out of the cv. The average velocity, \( \bar{V} = \frac{Q}{A} \).

For a steady, compressible flow,

\[
0 = \int_{cs} \rho \mathbf{V} \cdot \mathbf{n} \, dA_{cs} = \int_{cs} \rho \mathbf{V} \cdot dA_{cs} \quad (250)
\]

**EXAMPLE**  - Mass Flow through Multiport Device

Consider steady flow of water through the device shown in the diagram. The areas are: \( A_1 = 0.2 \, \text{ft}^2 \), \( A_2 = 0.5 \, \text{ft}^2 \), and \( A_3 = A_4 = 0.4 \, \text{ft}^2 \). The mass flow rate out through section 3 is given as 3.88 slug/sec. The volume flow rate in through section 4 is given as 1 ft³/sec, and \( V_1 = 10 \, \text{ft/sec} \). If properties are assumed uniform across all inlet and outlet flow sections, determine the flow velocity at section 2.
EXAMPLE PROBLEM

**GIVEN:** Steady flow of water through the device. Properties uniform at all ports.
\[ A_1 = 0.2 \text{ ft}^2 \quad A_2 = 0.5 \text{ ft}^2 \]
\[ A_3 = A_4 = 0.4 \text{ ft}^2 \quad \rho = 1.94 \text{ slug/ft}^3 \]
\[ \dot{m}_3 = 3.88 \text{ slug/sec (outflow)} \]
\[ \dot{V}_1 = 10\dot{V} \text{ ft/sec} \]
Volume flow rate in at 4: 1.0 \text{ ft}^3/\text{sec}

**FIND:** Velocity at section 2.

**SOLUTION:**
Choose a fixed control volume. Two possibilities are shown by dashed lines.

Basic equation:
\[ 0 = \frac{d}{dt} \int_{V_{CV}} \rho dV + \int_{S_{CS}} \rho \vec{V} \cdot d\vec{A} \]

Assumptions:
1. Steady flow
2. Incompressible flow
3. Uniform properties at each section where fluid crosses the CV boundaries

For steady flow, the first term is zero by definition, so
\[ 0 = \int_{S_{CS}} \rho \vec{V} \cdot d\vec{A} \]

In looking at either control volume, we see that there are four sections where mass flows across the control surface. Thus we write
\[ \int_{S_{CS}} \rho \vec{V} \cdot d\vec{A} = \int_{A_1} \rho \vec{V} \cdot d\vec{A} + \int_{A_2} \rho \vec{V} \cdot d\vec{A} + \int_{A_3} \rho \vec{V} \cdot d\vec{A} + \int_{A_4} \rho \vec{V} \cdot d\vec{A} = 0 \] (1)

Let us look at these integrals one at a time, recognizing that properties are uniform over each area and assuming that \( \rho \) = constant.
\[ \int_{A_1} \rho \vec{V} \cdot d\vec{A} = - \int_{A_1} |\rho V_1 A_1| \]
Sign of \( \vec{V} \cdot d\vec{A} \) is negative at surface 1.

With the absolute value signs indicated, we have accounted for the directions of \( \vec{V} \) and \( d\vec{A} \) in taking the dot product.

Since we do not know the direction of \( \vec{V}_3 \), we shall leave section 2 for the moment.
\[ \int_{A_3} \rho \vec{V} \cdot d\vec{A} = \int_{A_3} |\rho V_3 A_3| = |\rho V_3 A_3| \]
Sign of \( \vec{V} \cdot d\vec{A} \) is positive at surface 3, since flow is out.

\[ \int_{A_4} \rho \vec{V} \cdot d\vec{A} = - \int_{A_4} |\rho V_4 A_4| = - |\rho V_4 A_4| \]
Sign of \( \vec{V} \cdot d\vec{A} \) is negative at surface 4.

where \( Q \) is the volume flow rate.
From Eq. 1 above,

\[ \int_{A_2} \rho \vec{V} \cdot d \vec{A} = -\int_{A_1} \rho \vec{V} \cdot d \vec{A} - \int_{A_3} \rho \vec{V} \cdot d \vec{A} - \int_{A_4} \rho \vec{V} \cdot d \vec{A} \]

\[ = [\rho V_1 A_1] - \hat{n}_3 + \rho [\hat{Q}_d] \]

\[ = \left| \begin{array}{c} 1.94 \text{ slug} \times \frac{10 \text{ ft}}{\text{sec}} \times 0.2 \text{ ft}^2 \end{array} \right| - 3.88 \text{ slug} \times \frac{1.0 \text{ ft}^3}{\text{sec}} \]

\[ = 1.94 \text{ slug/sec} \]

\[ \int_{A_2} \rho \vec{V} \cdot d \vec{A} = 1.94 \text{ slug/sec} \]

Since this is positive, \( \vec{V} \cdot d \vec{A} \) at section 2 is positive. Flow is out, as shown in the sketch:

\[ \int_{A_2} \rho \vec{V} \cdot d \vec{A} = \int_{A_2} [\rho V \cdot dA] = |pV_2 A_2| = 1.94 \text{ slug/sec} \]

\[ |V_2| = \frac{1.94 \text{ slug/sec}}{\rho A_2} = \frac{1.94 \text{ slug}}{\text{sec}} \times \frac{\text{ft}^3}{1.94 \text{ slug}} \times \frac{1}{0.5 \text{ ft}^2} = 2 \text{ ft/sec} \]

Since \( V_2 \) is in the negative \( y \) direction, then

\[ \vec{V}_2 = -2 \hat{j} \text{ ft/sec} \]

This problem illustrates the procedure recommended for evaluating \( \int_{CS} \rho \vec{V} \cdot d \vec{A} \)