**Problem 1**

Applying Kirchoff’s current law, we get

\[ i = i_1 + i_2 + i_3 = C \frac{dV}{dt} + \frac{V}{R} + \frac{1}{L} \int V dt \]

Compare the above equation with the governing equation for the mass-spring-damper system:

\[ F = m \ddot{x} + b \dot{x} + kx = m \frac{dv}{dt} + bv + k v \int v dt \quad \text{where} \quad v = \text{velocity} \]

Therefore,

Force is electrical current and is the flow variable.

Velocity is the “electrical voltage” and is the effort variable.

Other parameters have the following analogies:

\[ m \rightarrow C; \quad b \rightarrow \frac{1}{R}; \quad k \rightarrow \frac{1}{L} \]

**Problem 2**

The lumped model of the micromechanical filter, under the assumptions stated, is shown below.

The electrical circuit representation for the device is shown below. **Series** analogy is used for equivalence between mechanical and electrical components. The coupling beam’s capacitor should have the difference of two currents flowing through the two resonant beams. This makes that capacitor to be in parallel as shown in.
Let us now see if this circuit represents our mechanical system correctly by writing down the governing dynamic equations. KVL for ABCDA gives

\[ \frac{F}{k} i_1 dt - bi_1 - \frac{di_1}{dt} - kc\{i_1 - i_2\} dt = 0 \]

KVL for BEFCB gives

\[ -m \frac{di_2}{dt} - bi_2 - k i_2 dt + F + \{i_1 - i_2\} dt = 0 \]

b

In these equations, by replacing \( i_1 \) with \( \dot{x}_1 \), \( i_2 \) with \( \dot{x}_2 \), we get

\[ m\ddot{x}_1 + b\dot{x}_1 + kx_1 + k_c (x_1 - x_2) = F \]

\[ m\ddot{x}_2 + b\dot{x}_2 + kx_2 - k_c (x_1 - x_2) = F \]

which, from the force-balance perspective, is the correct set of governing equations for the mechanical behavior.

**Problem 3**

KVL for ABEFA gives:
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\[ V = \frac{1}{C_i} \int i_i \, dt + L_i \frac{d(i_i - i_2)}{dt} + R i_i \]  \hspace{1cm} (1)

KVL for BCDEB gives:

\[ L_1 \frac{d(i_i - i_2)}{dt} = \frac{1}{C_2} \int i_2 \, dt + L_2 \frac{d i_2}{dt} \]  \hspace{1cm} (2)

Laplace transform of (1) and (2) can be written as

\[ V(s) = \frac{1}{C_i} \frac{I_i(s)}{s} + L_i s(I_i(s) - I_2(s)) + R I_i(s) \]  \hspace{1cm} (3)

\[ L_i s(I_i(s) - I_2(s)) = \frac{1}{C_2} \frac{I_2(s)}{s} + L_2 s I_2(s) \]  \hspace{1cm} (4)

Solve for \( I_2(s) \) in (4), and substitute in (3) to get

\[ \frac{I_i(s)}{V(s)} = \frac{s C_1 + C_1 C_2 (L_1 + L_2) s^3}{\{1 + L_1 C_i s^2 + RC_i s\} \{1 + C_2 (L_1 + L_2) s^2\} - C_1 C_2 L_i^2 s^4} \]

which can be simplified further to get the transfer function as

\[ \frac{I_i(s)}{V(s)} = \frac{C_1 C_2 (L_1 + L_2) s^3 + s C_i}{C_i C_2 L_1 L_2 s^4 + RC_i C_2 (L_1 + L_2) s^3 + \{C_2 (L_1 + L_2) + L_i C_i\} s^2 + RC_i s + 1} \]

**Problem 4**

There are four independent energy storage devices. Therefore, we define four state variables as

\[ x_1 = \int i_1 \, dt = Q_1 \]
\[ x_2 = \dot{x}_1 = i_1 \]
\[ x_3 = \int i_2 \, dt = Q_2 \]
\[ x_4 = \dot{x}_3 = i_2 \]

With this, equations (1) and (2) of the previous problem now become

\[ V = \frac{1}{C_i} x_1 + L_i (\dot{x}_2 - \dot{x}_4) + R x_2 \]
\[ L_1 (\ddot{x}_2 - \ddot{x}_4) = \frac{1}{C_2} x_3 + L_2 \ddot{x}_4 \]
Solve the above two equations for $\dot{x}_2$ and $\dot{x}_4$ treating everything else as constants to get

$$\dot{x}_2 = \frac{1}{L_2} \left( V - \frac{1}{C_1} x_1 - R x_2 - \frac{1}{C_2} x_3 \right)$$

$$\dot{x}_4 = \frac{1}{L_1} \left( V - \frac{1}{C_1} x_1 - R x_2 \right) + \frac{1}{2} \left( V - \frac{1}{C_1} x_1 - R x_2 - \frac{1}{C_2} x_3 \right)$$

The above two equations together with $\dot{x}_1 = x_2$ and $\dot{x}_3 = x_4$ are the state equations that we need.