Additional Problems for Assignment 10

Problem 1: Let $X$ be a discrete random variable following the geometric distribution with parameter $p$. Then:

$$ p_X(k) = P\{X = k\} = (1 - p)^{k-1}p, \quad k = 1, 2, \ldots $$

Show that the expected value and the variance of $X$ are:

$$ E[X] = \frac{1}{p}, \quad \text{Var}(X) = \frac{1-p}{p^2}. $$

Hint: Note that for any $x \in (-1, 1)$:

$$ \sum_{k=1}^{\infty} k x^{k-1} = \sum_{k=1}^{\infty} \frac{d}{dx}(x^k) = \frac{d}{dx} \left( \sum_{k=0}^{\infty} x^k \right) $$

$$ = \frac{d}{dx} \left( \frac{1}{1-x} \right) = \frac{1}{(1-x)^2}. $$

The same technique is used to calculate $\sum_{k=1}^{\infty} k^2 x^{k-1}$.

Problem 2: Let $X$ be a discrete random variable following the negative binomial distribution with parameters $(r, p)$:

$$ p_X(k) = P\{X = k\} = \binom{r-1}{k-1} p^r (1-p)^{k-r}, \quad k = r, r+1, \ldots $$

Show that the expected value and variance of $X$ are:

$$ E[X] = \frac{r}{p}, \quad \text{Var}(X) = \frac{r(1-p)}{p^2}. $$

Hint: Consider the interpretation of $X$ as the number of trials until the $r^{th}$ success in a series of independent trials with probability of success $p$. Let $X_1$
be the number of trials until the first success, \( X_2 \) the number of additional trials until the second success, and so on. Then:

\[
X = X_1 + X_2 + \cdots + X_r.
\]

Explain that \( X_1, \ldots, X_r \) are independent random variables following the geometric distribution with parameter \( p \).

**Problem 3:** An urn contains \( r \) red and \( w \) white balls. We draw \( n \) balls at random. Let \( X \) denote the number of red balls in that random sample. \( X \) follows the hypergeometric distribution:

\[
p_X(k) = P\{X = k\} = \frac{\binom{r}{k} \binom{w}{n-k}}{\binom{N}{n}}, \quad k = 0, 1, \ldots, r,
\]

where \( N = r + w \).

1. Show that the following identity holds for any integers \( n, k \):

\[
k \binom{n}{k} = n \binom{n-1}{k-1}.
\]

2. Show that the expected value of \( X \) is:

\[
E[X] = \frac{nr}{N}.
\]

**Hint:** Use the identity from part (1) to write:

\[
k \binom{r}{k} = r \binom{r-1}{k-1} \quad \text{and} \quad n \binom{N}{n} = N \binom{N-1}{n-1}.
\]

**Problem 4:** Let \( X \) be a continuous random variable uniformly distributed over interval \((a, b)\). Compute \( E[X] \) and \( \text{Var}(X) \).

**Problem 5:** Let \( X \) be a continuous random variable following the exponential distribution with parameter \( \lambda \):

\[
f_X(x) = \lambda e^{-\lambda x}, \quad x > 0.
\]

Show that:

\[
E[X] = \frac{1}{\lambda}, \quad \text{Var}(X) = \frac{1}{\lambda^2}.
\]