Digital Channels

At a suitable level of abstraction, the communication process may be viewed as being accomplished over a digital channel which sits atop the underlying analogue channel. To a certain extent, this just means that the physical processes of modulation and demodulation of the transmitted analogue signals are transparent to the user at this level of abstraction. Marked benefits can accrue, however, if the physical channels are utilised to directly accommodate digital data, as in direct encoding (bit-by-bit) digital signalling methods such as the NRZ signalling strategy and its bipolar and biphase variants. In such situations, the data link control unit can interface almost directly with the underlying physical channel (instead of through a modem). The simplicity of the interaction between the data link control layer and the physical layer provides the key to the achievement of higher data rates at lower probabilities of error than would be possible in an analogue channel.

Repeaters

It is difficult to combat signal attenuation in an analogue channel as amplification of the signal amplifies the noise as well, with the noise level building up with every stage of amplification. In a digital channel, however, the signal can be intercepted at a repeater before it gets too severely corrupted by noise and intersymbol interference and the transmitted bits recovered; a noise-free digital signal can now be regenerated at the repeater and sent on down the channel. The functioning of a repeater in a digital channel is hence intrinsically non-linear. As long as the signal entering the repeater is not too severely corrupted, the receiver will effectively wipe out the noise and transmit a clean regenerated signal anew.

To more clearly appreciate the noise suppression properties of the repeaters, consider a digital communication link with \( n \) repeater stages, as shown.

In this setup, a bit \( b = b^{(i)} \in \{0, 1\} \) that is originally transmitted is regenerated \( n \) times as it passes through \( n \) repeater stages with \( b^{(i)} \in \{0, 1\} \) the bit estimate after the \( i \)th stage.

For \( 1 \leq i \leq n \), noisy bit transmission between repeater stages \( (i-1) \) and \( i \) may be modelled by a binary symmetric channel (BSC) with raw bit error probability \( p_i = \Pr\{b^{(i)} \neq b^{(i-1)}\} \). Alternatively, in modulo 2 arithmetic, let \( S_i = b^{(1)} + b^{(i-1)} \mod 2 \). Then \( S_i \sim \text{Bernoulli}(p_i) \) is a Bernoulli random
variable with success probability $p_i$:  

$$S_i = \begin{cases} 
0 & \text{with probability } q_i = 1 - p_i, \\
1 & \text{with probability } p_i. 
\end{cases}$$

Observe that we may suppose, without loss of generality, that $p_i \leq 1/2$. (Why?) If we suppose additionally that the receivers are equispaced, then we may make the simplifying assumption that the link segments between repeaters are all statistically identical, i.e., are modelled as BSCs with the same raw bit error probability $p_i = p$ ($1 \leq i \leq n$).

In addition, to further simplify the analysis, suppose that transmissions over the different stages are statistically independent of each other, i.e., $\{S_i, 1 \leq i \leq n\}$ is a collection of independent, identically distributed, Bernoulli random variables with success probability $p$. Each of the BSCs may be viewed schematically as in the binary channel shown alongside. Under these conditions, what is the end-to-end bit error probability $P_{\text{bit}} = P_{\text{bit}}^{(n)} = \Pr\{b^{(n)} \neq b\}$ that a transmitted bit $b$ will be received in error after $n$ repeater stages?

We consider two different approaches to the problem. The different analytical approaches are instructive in their own right and, as we shall see, of use subsequently.

**Inductive Approach** We begin by setting up a recurrence for $P_{\text{bit}}^{(n)}$. A given bit will be in error after $n$ stages if either the bit is in error after $n-1$ stages and the erroneous bit after $n-1$ stages is transmitted error-free over the $n$th stage, or the bit is not in error after $n-1$ stages and is erroneously transmitted over the $n$th stage. We consequently obtain the recurrence 

$$P_{\text{bit}}^{(n)} = P_{\text{bit}}^{(n-1)}(1-p) + (1-P_{\text{bit}}^{(n-1)})p = p + (1-2p)P_{\text{bit}}^{(n-1)} \quad (*)$$

valid for all $n \geq 2$. As $n$ is a generic variable, we could replace $n$ by $n-1$, $n-2$, $\ldots$, $2$, in turn above to obtain

$$\begin{align*}
P_{\text{bit}}^{(n-1)} &= p + (1-2p)P_{\text{bit}}^{(n-2)}, \\
P_{\text{bit}}^{(n-2)} &= p + (1-2p)P_{\text{bit}}^{(n-3)}, \\
& \vdots \\
P_{\text{bit}}^{(2)} &= p + (1-2p)P_{\text{bit}}^{(1)}. 
\end{align*}$$

We may now substitute in turn for the quantities on the right-hand side
of (**) to obtain the sequence of equations
\[
P_{\text{bit}}^{(n)} = p(1 - 2p)^0 + (1 - 2p)^1 p_{\text{bit}}^{(n-1)}
\]
\[
= p(1 - 2p)^0 + p(1 - 2p)^1 + (1 - 2p)^2 p_{\text{bit}}^{(n-2)}
\]
\[
= p(1 - 2p)^0 + p(1 - 2p)^1 + (1 - 2p)^2 p_{\text{bit}}^{(n-3)}
\]
\[
= p(1 - 2p)^0 + p(1 - 2p)^1 + \cdots + p(1 - 2p)^n - 1 + (1 - 2p)^n p_{\text{bit}}^{(1)},
\]
where the final expression may be readily verified by induction. Now observe that
\[
P_{\text{bit}}^{(1)} = p
\]
is simply the crossover probability of a single BSC. It follows that
\[
P_{\text{bit}}^{(n)} = p \sum_{i=0}^{n-1} (1 - 2p)^i.
\]
We recognise the sum as just a finite geometric series. Recall that
\[
\sum_{i=0}^{n-1} x^i = \frac{1 - x^n}{1 - x}, \quad (|x| < 1).
\]
Identifying 1 - 2p with x we obtain
\[
P_{\text{bit}}^{(n)} = p \left( \frac{1 - (1 - 2p)^n}{1 - (1 - 2p)} \right) = \frac{1}{2} (1 - (1 - 2p)^n)
\]
for all \( n \geq 1 \). The final closed form result for the end-to-end bit error probability is remarkable in its simplicity and suggests that there may be a direct approach to the solution. Indeed there is.

Direct Approach. A bit error will transpire after \( n \) stages if, and only if, there is an error in transmission over an odd number of stages. The probability that exactly \( i \) stages had errors in transmission is easily seen to be given by the binomial \( \binom{n}{i} p^i (1 - p)^{n-i} \). Consequently,
\[
P_{\text{bit}}^{(n)} = \sum_{i \text{ odd}} \binom{n}{i} p^i (1 - p)^{n-i}.
\]
The sum on the right-hand side is easily evaluated via the binomial theorem:
\[
(x + y)^n = \sum_{i=0}^{n} \binom{n}{i} x^i y^{n-i},
\]
where the binomial coefficients \( \binom{n}{i} \) may be defined recursively via Pascal’s triangle
\[
\binom{n}{i} = \binom{n-1}{i-1} + \binom{n-1}{i}
\]
or, more directly, as

\[
\binom{n}{i} = \frac{n!}{i!(n-i)!}.
\]

Now write \( q = 1 - p \) for simplicity and observe that

\[
(q + p)^n = \sum_{i=0}^{n} \binom{n}{i} p^i q^{n-i} = \sum_{i \text{ even}} \binom{n}{i} p^i q^{n-i} + \sum_{i \text{ odd}} \binom{n}{i} p^i q^{n-i},
\]

\[
(q - p)^n = \sum_{i=0}^{n} \binom{n}{i} (-p)^i q^{n-i} = \sum_{i \text{ even}} \binom{n}{i} p^i q^{n-i} - \sum_{i \text{ odd}} \binom{n}{i} p^i q^{n-i}.
\]

By subtracting the second line from the first, it follows immediately that

\[
P_{\text{bit}}(n) = \frac{1}{2} \left( (q + p)^n - (q - p)^n \right) = \frac{1}{2} (1 - 1 - 2p)^n,
\]

as obtained by the indirect inductive approach.

Suppose that the communication link has total length \( L \) and that \( n \) repeater stages are scattered uniformly through the link with a spacing of \( l = L/n \) between repeaters. In general, the binary symmetric channel modelling the link between any two repeaters has a raw bit error probability \( p = p(l) = p(L/n) \) which increases monotonically with repeater separation \( l \). The end-to-end bit error probability is then given by

\[
\Pr(\text{bit error over communication link}) = P_{\text{bit},L}(n) = \frac{1}{2} \left( 1 - 1 - 2p(L/n) \right)^n.
\]

For instance, if the raw BSC bit error probability \( p(l) = c l \) increases linearly with distance (for sufficiently small distances), then, in the limit of a large number of repeaters,

\[
P_{\text{bit},L}(n) \rightarrow P_{\text{bit},L} = \frac{1}{2} \left( 1 - e^{-2cL} \right) \quad (n \rightarrow \infty).
\]

If the raw bit error probability increases superlinearly with \( l \) (i.e., \( p(l)/l \rightarrow 0 \) as \( l \rightarrow 0 \)) then, a fortiori, \( P_{\text{bit},L}(n) \rightarrow 0 \) as \( n \rightarrow \infty \). Indeed, it can be seen with a modicum of effort that \( P_{\text{bit},L}(n) \) decreases monotonically with the number of repeaters \( n \) if the raw bit error probability \( p(l) \) of the constituent BSCs increases at least linearly with repeater separation \( l \). This is the case for using many repeaters in practice.

**Time Division Multiplexing (TDM)**

As we saw earlier, a communication link with sufficient bandwidth can be shared among several users by multiplexing the users in frequency. A completely analogous sharing of common resources may be effected in the time
domain in a packet switched environment by cycling through users in a round robin fashion in time. In a typical scenario, data are sent in successive frames. Each frame consists of a fixed number, say $m$, of equal sized slots, one for each data stream to be multiplexed, together with a few bits for control and synchronisation purposes.

Viewed in time, transmission progresses sequentially, frame by frame; each user is visited in round robin fashion with data from the $i$th user transmitted whenever the $i$th slot in a given frame comes up for transmission. The number of bits from each data stream transmitted in each frame is typically between one to eight bits depending on the slot size in the frame. The illustrative frame of 17 bits shown above consists of four slots of four bits apiece, together with a synchronisation bit.

This form of data multiplexing is called time division multiplexing (or simply, TDM). The main feature of TDM is to make available to each of $m$ users a guaranteed slot in which data can be transmitted so that the delay is fixed. This is of importance in applications like telephony where variance in packet delay leads to unacceptable distortions in speech. From an abstract point of view, we can think of TDM as a technique for splitting a channel into $m$ independent sub-channels, one for each user. In this viewpoint, the individual sub-channels can each accommodate about $1/m$ of the traffic that the single channel can. Examples of TDM systems are the T1 and T3 carrier systems described below.

Example 1 Digital Telephony.

The T1 carrier system is a time division multiplexed system designed to accommodate 24 voice channels, primarily for use in heavy traffic metropolitan areas. Each frame consists of 193 bits of which the first is a synchronisation bit with the remaining 192 bits divided into 24 slots of eight bits each. As each voice channel requires a throughput of 64 kbps, the T1 carrier maintains an overall data rate of 1.544 Mbps.

The T1 system has been widely adopted for use throughout the United States, Canada, and Japan. A similar system adopted for use in Europe multiplexes 32 data streams at an overall data rate of 2.048 Mbps.

Several T1 carriers may be multiplexed in time to serve a larger geographical area. The T3 carrier system, for instance, multiplexes 28 T1 carriers at an overall data rate of 44.736 Mbps.
Statistical Multiplexing

A drawback of the fixed slot assignments in TDM is that if packets from a user are temporarily in abeyance, then the slots assigned to that user will go unused until data from the user are again available. Thus, the channel may not be utilised optimally resulting in lower data throughput than potentially achievable. An alternative idea is to use statistical multiplexing in which arriving data from all users are buffered by a data concentrator and sent over the channel in accordance with some protocol, for instance, a first-in, first-out (FIFO) protocol. Statistical multiplexing will guarantee that the channel will never be idle if there are data to be transmitted thus maximising the overall data throughput (or, equivalently, minimising the overall mean delay). However, the lack of a guaranteed, periodic slot for each user means that packets from any given user see variable delays depending on the size of the buffer queue seen by each incoming packet. While this variance in delay is unacceptable for speech, it is of no account in data transmissions. In such applications, hence, statistical multiplexing would be preferred over time or frequency division multiplexing as making optimal use of channel resources to maximise overall data throughput.

Integrated Services Digital Networks (ISDN)

The telephone network is comprised generically of two parts: the local loops which go from users to local offices, and the internal networks connecting local and central offices and toll switches. The internal networks are now almost exclusively digital using time division multiplexed systems such as the T1 and T3 carrier systems which offer 64 kbps communication to each user. The bottleneck in data communication lies in the local loops where the communication is still largely over analogue channels. Digitising the local loops would allow the user 64 kbps digital service (or faster) over a single voice circuit. Such an integration of voice and data services is called an integrated services digital network (or ISDN, for short).

A standard ISDN service, for instance, makes available two 64 kbps channels together with a single 16 kbps channel to the user. Each of the 64 kbps channels could be used as a voice channel or as a data channel while the 16 kbps channel could be used to control the two 64 kbps channels, or even for low data rate applications such as alarm and security services.

The proposed digital subscriber loops (or simply, DSL) will provide digital service to subscribers at much higher rates over existing twisted wire pairs. However, substantial infrastructure work—such as the installation of a large number of repeaters and signal encoders and decoders—needs to be done to convert the local loops into digital channels without extensive rewiring.
Replacement of twisted wire pairs in the local loops by optical fibre will allow much higher data rates with very low error rates leading to what is called broadband integrated services digital networks (BISDN). The standard interfaces for these high-speed optical links are called synchronous optical networks (or SONET). Thus, for instance, the SONET STS-3 optical link allows a standard user access rate of 155 Mbps. At these rates, video conferencing and real-time, high definition image transmission become accessible.