TCOM501 – Networking: Theory & Fundamentals
Final Examination
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Answer all problems. Good Luck!

Problem 1 [50 points]: Two types of jobs arrive at a service station according to two independent Poisson processes with rates $\lambda_1 = 4$ jobs/s and $\lambda_2 = 1$ job/s, respectively. The service times are independent of each other and of the arrival processes. The service times of type-1 jobs are exponentially distributed with mean 0.1 s. The service times of type-2 jobs are exponentially distributed with mean 0.2 s. The service station has a single server and an infinite storage buffer.

We consider the following configurations of the service station:

Configuration 1: All jobs are merged into a common queue and served by the single server on a First-Come-First-Served basis.

Configuration 2: Non-preemptive priority queueing is implemented, where type-1 jobs have priority over type-2 jobs.

Configuration 3: Non-preemptive priority queueing is implemented, where type-2 jobs have priority over type-1 jobs.

Configuration 4: There is a separate queue dedicated to each type of job. A fixed portion $a_1$ of the server’s capacity is allocated to the queue of type-1 jobs, and the remaining portion $1-a_1$ is allocated to the queue of type-2 jobs.

Part A

For a given configuration, let $W_1$ and $W_2$ denote the average waiting time of jobs of type-1 and type-2, respectively; let $W$ denote the average waiting time of a “typical” job (independent of type).

1. [10 points] Consider configurations 1, 2, and 3:
   a. Find $W_1$, $W_2$, and $W$ for each configuration.
   b. Compare the overall waiting times ($W$) of the three configurations and provide an intuitive explanation of your observations.

2. [10 points] Consider configuration 4:
   a. Express $W_1$, $W_2$, and $W$ in terms of $a_1$. For what values of $a_1$ is the system stable?
   b. Find the value of $a_1$ that minimizes the overall waiting time $W$.
   c. Compare the minimal value of the overall waiting time with the overall waiting times of the other configurations and provide an intuitive explanation of your observations.

Part B

3. [20 points] For configuration 1, find the probability $p_n$, that at steady-state the total number of jobs in the system is $n = 0, 1, \ldots$

4. [10 points] Consider configuration 4 and assume $a_1=1/2$. Find the probability $p_n$, that at steady-state the total number of jobs in the system is $n = 0, 1, \ldots$
Problem 2 [20 points]: Consider the network of Figure 1, which is represented by a directional graph, with cost (length) \( d_{ij} \) associated with each directed link \((i,j)\).

![Figure 1](image)

1. [7 points] Find the shortest distance from every node in the network to destination node 1, and the corresponding shortest path tree using the Bellman-Ford Algorithm. Indicate clearly the operations performed at each iteration of the algorithm.

2. [7 points] Repeat part 1 using the Dijkstra algorithm.

3. [6 points] Consider the network obtained by replacing each pair of directed links \((i,j)\) and \((j,i)\) with a bidirectional link with weight \( w_{ij} = \min\{d_{ij}, d_{ji}\} \); if directed link \((i,j)\) does not exist its cost is \( d_{ij} = \infty \). We want to find a minimum weight spanning tree (MST). Since the link weights are not distinct, there might be multiple spanning trees with minimum weight. However, if we define a rule that brakes the ties when links with the same weight are compared, any iterative algorithm will construct the same MST.
   a. Consider two links \((i,j)\) and \((k,l)\) with equal weights. Define a rule that determines in a unique way the link that will be chosen by an MST construction algorithm.
   b. Applying the rule you proposed in part 3a, find the unique MST using the Prim-Dijkstra algorithm. Indicate clearly all steps in the construction of the MST.
      [If you did not answer part 3a, just use the algorithm to find a MST.]
   c. Repeat part 3b, using Kruskal’s algorithm.
Problem 3 [15 points]: Consider a closed Jackson network with $K$ nodes and $M$ customers. The normalization constant for the network is:

$$G(M) = \sum_{n_1,\ldots,n_K=M} \rho_1^{n_1} \cdots \rho_K^{n_K}$$

(1)

1. [3 points] Let $x_i$ be the number of customers at node $i$ at steady state. Prove that:

$$P\{x_i \geq m\} = \rho_i^m \frac{G(M-m)}{G(M)}, \quad m = 1, 2, \ldots, M$$

2. [2 points] Prove that the average throughput of node $i$ is:

$$\gamma_i(M) = \lambda_i \frac{G(M-1)}{G(M)}$$

(2)

3. [10 points] Mean Value Analysis: Define for all nodes $i$, and for $m = 1, 2, \ldots, M$:

- $N_i(m)$: average number of customers at node $i$
- $T_i(m)$: average time that a customer spends (per visit) at node $i$
- $\gamma_i(m)$: average throughput of node $i$

when the total number of customers in the closed network is $m$. Show that, for all nodes $i$, $N_i(M)$ and $T_i(M)$ can be calculated iteratively as:

$$T_i(m) = \frac{1 + N_i(m-1)}{\mu_i}, \quad m = 1, \ldots, M$$

(3)

$$N_i(m) = m \frac{\lambda_i T_i(m)}{\sum_{j=1}^K \lambda_j T_j(m)}, \quad m = 1, \ldots, M$$

(4)

with initial condition:

$$N_i(0) = 0$$

Problem 4 [15 points]: Consider the closed Jackson network of Figure 2, with $K=3$ nodes and $M$ customers.

![Figure 2](image)

1. [5 points] Calculate the normalization constant $G(M)$ using eq. (1).
2. [5 points] For each node $i$, find as functions of the number of customers $M$:
   - the average throughput $\gamma_i(M)$
   - the average number of customers $N_i(M)$

3. [5 points] Using Mean Value Analysis described by eqs. (3) and (4), find for each node $i$, $T_i(M)$, $N_i(M)$, and $\gamma_i(M)$ for $M = 1, 2, 3, 4, \text{and } 5$.

**Useful Formulas**

**M/M/1 Queue**

Stationary distribution: $p_n = (1-\rho)^n \rho^n$, $\rho = \lambda / \mu$

Average time a customer spends in the system (including service): $T = \frac{1}{\mu - \lambda}$

**M/G/1 Queue**

Pollaczeck-Khinchin formula: $W = \frac{\lambda E[X^2]}{2(1-\rho)}, \quad \rho = \lambda E[X]$

Pollaczeck-Khinchin transform equation: the z-transform of the stationary distribution is

$$G_N(z) = \sum_{n=0}^{\infty} p_n z^n = \frac{(1-\rho)(z-1)M_X(\lambda(z-1))}{z - M_X(\lambda(z-1))}$$

where $M_X(t) = E[e^{tX}]$ is the moment generating function of the service time $X$.

**Non-preemptive Priority Queueing**

$$W_i = \frac{R}{1-\rho_i}$$

$$W_k = \frac{R}{(1-\rho_1-\cdots-\rho_{k-1})(1-\rho_1-\cdots-\rho_{k-1}-\rho_k)}, \quad k \geq 2$$

where $R = \frac{1}{2} \sum_{i=1}^{k} \lambda_i E[X_i^2]$.

**Bellman-Ford Algorithm:**

$$D_i^{k+1} = \min_{j} \{ d_{ij} + D_j^k \}, \quad i \neq 1$$

$$D_1^{k+1} = 0$$

with initial conditions:
\[ D_i^0 = \infty, \quad i \neq 1 \]
\[ D_1^0 = 0 \]

**Dijkstra’s Algorithm:**

*Initialization:* \( P = \{1\}, \quad D_1 = 0, \quad D_j = d_{j1}, \forall j \neq 1 \)

*Iteration step 1:* Find \( i \not\in P \), such that \( D_i = \min_{j \in P} D_j \). Set \( P := P \cup \{i\} \).

*Iteration step 2:* For all \( j \not\in P \), set: \( D_j := \min \{D_j, d_{ji} + D_i\} \).