Topics

- Markov Chains
- Discrete-Time Markov Chains
- Calculating Stationary Distribution
- Global Balance Equations
- Detailed Balance Equations
- Birth-Death Process
- Generalized Markov Chains
- Continuous-Time Markov Chains
Markov Chain

- Stochastic process that takes values in a countable set
  - Example: \{0,1,2,...,m\}, or \{0,1,2,...\}
  - Elements represent possible “states”
  - Chain “jumps” from state to state

- Memoryless (Markov) Property: Given the present state, future jumps of the chain are independent of past history

- Markov Chains: discrete- or continuous- time
Discrete-Time Markov Chain

- Discrete-time stochastic process \( \{X_n: n = 0, 1, 2, \ldots \} \)
- Takes values in \( \{0, 1, 2, \ldots \} \)
- Memoryless property:
  \[
  P(\{X_{n+1} = j \mid X_n = i, X_{n-1} = i_{n-1}, \ldots, X_0 = i_0\}) = P(\{X_{n+1} = j \mid X_n = i\})
  \]
  \[
  P_{ij} = P(\{X_{n+1} = j \mid X_n = i\})
  \]
- Transition probabilities \( P_{ij} \)
  \[
  P_{ij} \geq 0, \quad \sum_{j=0}^{\infty} P_{ij} = 1
  \]
- Transition probability matrix \( P = [P_{ij}] \)
Chapman-Kolmogorov Equations

- $n$ step transition probabilities

\[ P^n_{ij} = P\{X_{n+m} = j \mid X_m = i\}, \quad n, m \geq 0, \ i, j \geq 0 \]

- Chapman-Kolmogorov equations

\[ P^{n+m}_{ij} = \sum_{k=0}^{\infty} P^n_{ik} P^m_{kj}, \quad n, m \geq 0, \ i, j \geq 0 \]

- $P^n_{ij}$ is element $(i, j)$ in matrix $P^n$

- Recursive computation of state probabilities
State Probabilities – Stationary Distribution

- State probabilities (time-dependent)
  \[ \pi_j^n = P\{X_n = j\}, \quad \pi^n = (\pi_0^n, \pi_1^n, \ldots) \]
  \[ P\{X_n = j\} = \sum_{i=0}^{\infty} P\{X_{n-1} = i\}P\{X_n = j \mid X_{n-1} = i\} \Rightarrow \pi_j^n = \sum_{i=0}^{\infty} \pi_i^{n-1}P_{ij} \]

  In matrix form:
  \[ \pi^n = \pi^{n-1}P = \pi^{n-2}P^2 = \ldots = \pi^0P^n \]

- If time-dependent distribution converges to a limit
  \[ \pi = \lim_{n \to \infty} \pi^n \]
  \[ \pi \] is called the *stationary distribution*

  \[ \pi = \pi P \]

  > Existence depends on the structure of Markov chain
Classification of Markov Chains

Irreducible:
- States i and j communicate:
  \[ \exists n, m : P_{ij}^n > 0, P_{ji}^m > 0 \]
- Irreducible Markov chain: all states communicate

Aperiodic:
- State i is periodic:
  \[ \exists d > 1 : P_{ii}^n > 0 \Rightarrow n = \alpha d \]
- Aperiodic Markov chain: none of the states is periodic
Theorem 1: Irreducible aperiodic Markov chain

- For every state \( j \), the following limit exists and is independent of initial state \( i \)

\[
\pi_j = \lim_{n \to \infty} P\{X_n = j \mid X_0 = i\}, \quad i = 0, 1, 2, ...
\]

- \( N_j(k) \): number of visits to state \( j \) up to time \( k \)

\[
P\left\{ \pi_j = \lim_{k \to \infty} \frac{N_j(k)}{k} \mid X_0 = i \right\} = 1
\]

- \( \pi_j \): frequency the process visits state \( j \)
Existence of Stationary Distribution

**Theorem 2:** Irreducible aperiodic Markov chain. There are two possibilities for scalars:

\[
\pi_j = \lim_{n \to \infty} P\{X_n = j \mid X_0 = i\} = \lim_{n \to \infty} P_{ij}^n
\]

1. \( \pi_j = 0 \), for all states \( j \) \( \Rightarrow \) No stationary distribution
2. \( \pi_j > 0 \), for all states \( j \) \( \Rightarrow \) \( \pi \) is the *unique* stationary distribution

**Remark:** If the number of states is finite, case 2 is the only possibility
Ergodic Markov Chains

- Markov chain with a stationary distribution
  \[ \pi_j > 0, \quad j = 0, 1, 2, \ldots \]

- States are positive recurrent: The process returns to state j "infinitely often"

- A positive recurrent and aperiodic Markov chain is called ergodic

- Ergodic chains have a unique stationary distribution
  \[ \pi_j = \lim_{n \to \infty} P^n_{ij} \]

- Ergodicity ⇒ Time Averages = Stochastic Averages
Calculation of Stationary Distribution

A. Finite number of states
- Solve explicitly the system of equations
  \[ \pi_j = \sum_{i=0}^{m} \pi_i P_{ij}, \quad j = 0, 1, \ldots, m \]
  \[ \sum_{i=0}^{m} \pi_i = 1 \]
- Numerically from \( P^n \) which converges to a matrix with rows equal to \( \pi \)
- Suitable for a small number of states

B. Infinite number of states
- Cannot apply previous methods to problem of infinite dimension
- Guess a solution to recurrence:
  \[ \pi_j = \sum_{i=0}^{\infty} \pi_i P_{ij}, \quad j = 0, 1, \ldots, \]
  \[ \sum_{i=0}^{\infty} \pi_i = 1 \]
Example: Finite Markov Chain

Absent-minded professor uses two umbrellas when commuting between home and office. If it rains and an umbrella is available at her location, she takes it. If it does not rain, she always forgets to take an umbrella. Let \( p \) be the probability of rain each time she commutes. What is the probability that she gets wet on any given day?

- Markov chain formulation
- \( i \) is the number of umbrellas available at her current location
- Transition matrix

\[
P = \begin{bmatrix}
0 & 0 & 1 \\
0 & 1-p & p \\
1-p & p & 0
\end{bmatrix}
\]
Example: Finite Markov Chain

\[ P = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1-p & p \\ 1-p & p & 0 \end{bmatrix} \]

\[ \begin{align*}
\begin{cases}
\pi = \pi P \\
\sum_i \pi_i = 1
\end{cases}
\quad \iff 
\begin{cases}
\pi_0 = (1-p)\pi_2 \\
\pi_1 = (1-p)\pi_1 + p\pi_2 \\
\pi_2 = \pi_0 + p\pi_1 \\
\pi_0 + \pi_1 + \pi_2 = 1
\end{cases}
\quad \iff 
\pi_0 = \frac{1-p}{3-p}, \pi_1 = \frac{1}{3-p}, \pi_2 = \frac{1}{3-p}
\end{align*} \]

\[ P\{\text{gets wet}\} = \pi_0 p = p \frac{1-p}{3-p} \]
Example: Finite Markov Chain

- Taking $p = 0.1$:

  \[
  \pi = \left( \frac{1-p}{3-p}, \frac{1}{3-p}, \frac{1}{3-p} \right) = (0.310, 0.345, 0.345)
  \]

  \[
  P = \begin{bmatrix}
  0 & 0 & 1 \\
  0 & 0.9 & 0.1 \\
  0.9 & 0.1 & 0
  \end{bmatrix}
  \]

- Numerically determine limit of $P^n$

  \[
  \lim_{n \to \infty} P^n = \begin{bmatrix}
  0.310 & 0.345 & 0.345 \\
  0.310 & 0.345 & 0.345 \\
  0.310 & 0.345 & 0.345
  \end{bmatrix} \quad (n \approx 150)
  \]

- Effectiveness depends on structure of $P$
Global Balance Equations

- Markov chain with infinite number of states
- Global Balance Equations (GBE)

\[ \pi_j \sum_{i=0}^{\infty} P_{ji} = \sum_{i=0}^{\infty} \pi_i P_{ij} \Leftrightarrow \pi_j \sum_{i \neq j} P_{ji} = \sum_{i \neq j} \pi_i P_{ij}, \quad j \geq 0 \]

- \( \pi_j P_{ji} \) is the frequency of transitions from \( j \) to \( i \)

\[
\begin{pmatrix}
\text{Frequency of transitions out of } j \\
\text{Frequency of transitions into } j
\end{pmatrix} =
\begin{pmatrix}
\text{Frequency of transitions out of } j \\
\text{Frequency of transitions into } j
\end{pmatrix}
\]

- Intuition: \( j \) visited infinitely often; for each transition out of \( j \) there must be a subsequent transition into \( j \) with probability 1
Global Balance Equations

- Alternative Form of GBE

\[
\sum_{j \in S} \pi_j \sum_{i \notin S} P_{ji} = \sum_{i \notin S} \pi_i \sum_{j \in S} P_{ij}, \quad S \subseteq \{0, 1, 2,...\}
\]

- If a probability distribution satisfies the GBE, then it is the unique stationary distribution of the Markov chain.

- Finding the stationary distribution:
  - Guess distribution from properties of the system
  - Verify that it satisfies the GBE

😊 Special structure of the Markov chain simplifies task
Global Balance Equations – Proof

\[ \pi_j = \sum_{i=0}^{\infty} \pi_i P_{ij} \quad \text{and} \quad \sum_{i=0}^{\infty} P_{ji} = 1 \quad \Rightarrow \]

\[ \pi_j \sum_{i=0}^{\infty} P_{ji} = \sum_{i=0}^{\infty} \pi_i P_{ij} \quad \Leftrightarrow \quad \pi_j \sum_{i \neq j} P_{ji} = \sum_{i \neq j} \pi_i P_{ij} \]

\[ \pi_j \sum_{i=0}^{\infty} P_{ji} = \sum_{i=0}^{\infty} \pi_i P_{ij} \quad \Rightarrow \quad \sum_{j \in S} \pi_j \sum_{i \in S} P_{ji} + \sum_{i \notin S} P_{ji} = \sum_{j \in S} \left( \sum_{i \in S} \pi_i P_{ij} + \sum_{i \notin S} \pi_i P_{ij} \right) \quad \Rightarrow \]

\[ \sum_{j \in S} \pi_j \sum_{i \notin S} P_{ji} = \sum_{i \notin S} \pi_i \sum_{j \in S} P_{ij} \]
Birth-Death Process

- One-dimensional Markov chain with transitions only between neighboring states: \( P_{ij} = 0 \), if \(|i-j| > 1\)
- Detailed Balance Equations (DBE)
  \[
  \pi_n P_{n,n+1} = \pi_{n+1} P_{n+1,n} \quad n = 0, 1, \ldots
  \]
- Proof: GBE with \( S = \{0, 1, \ldots, n\} \) give:
  \[
  \sum_{j=0}^{n} \sum_{i=n+1}^{\infty} \pi_j P_{ji} = \sum_{j=0}^{n} \sum_{i=n+1}^{\infty} \pi_i P_{ij} \Rightarrow \pi_n P_{n,n+1} = \pi_{n+1} P_{n+1,n}
  \]
Example: Discrete-Time Queue

- In a time-slot, one arrival with probability $p$ or zero arrivals with probability $1-p$
- In a time-slot, the customer in service departs with probability $q$ or stays with probability $1-q$
- Independent arrivals and service times
- State: number of customers in system

State Transition Diagram:

- State 0:
  - Arrival: $p$ with probability $q(1-p)$
  - Departure: $q$ with probability $(1-p)$

- State 1:
  - Arrival: $p$ with probability $p(1-q)$
  - Departure: $p$ with probability $(1-p)(1-q) + pq$

- State 2:
  - Arrival: $p$ with probability $q(1-p)$
  - Departure: $p$ with probability $(1-p)(1-q) + pq$

- State $n$:
  - Arrival: $p$ with probability $q(1-p)$
  - Departure: $p$ with probability $(1-p)(1-q) + pq$

- State $n+1$:
  - Arrival: $p$ with probability $q(1-p)$
  - Departure: $p$ with probability $(1-p)(1-q) + pq$
Example: Discrete-Time Queue

\[
\begin{align*}
\pi_0 p &= \pi_1 q(1 - p) \Rightarrow \pi_1 = \frac{p}{1 - p} \pi_0 \\
\pi_n p(1 - q) &= \pi_{n+1} q(1 - p) \Rightarrow \pi_{n+1} = \frac{p(1 - q)}{q(1 - p)} \pi_n, \quad n \geq 1
\end{align*}
\]

Define: \( \rho \equiv \frac{p}{q}, \ \alpha \equiv \frac{p(1 - q)}{q(1 - p)} \)

\[
\begin{cases}
\pi_1 = \frac{\rho}{1 - p} \pi_0 \\
\pi_n = \alpha^{n-1} \frac{\rho}{1 - p} \pi_0, \quad n \geq 1 \\
\pi_{n+1} = \alpha \pi_n, \quad n \geq 1
\end{cases}
\]
Example: Discrete-Time Queue

- Have determined the distribution as a function of $\pi_0$

$$\pi_n = \alpha^{n-1} \frac{\rho}{1-p} \pi_0, \; n \geq 1$$

- How do we calculate the normalization constant $\pi_0$?

- Probability conservation law:

$$\sum_{n=0}^{\infty} \pi_n = 1 \Rightarrow \pi_0 = \left[1 + \frac{\rho}{1-p} \sum_{n=1}^{\infty} \alpha^{n-1}\right]^{-1} = \left[1 + \frac{\rho}{(1-p)(1-\alpha)}\right]^{-1}$$

- Noting that

$$(1-p)(1-\alpha) = (1-p)\frac{q(1-p) - p(1-q)}{q(1-p)} = \frac{q-p}{q} = 1-\rho$$

$$\begin{cases} \pi_0 = 1-\rho \\ \pi_n = \rho(1-\alpha)\alpha^{n-1}, \; n \geq 1 \end{cases}$$
Detailed Balance Equations

- General case:
  \[ \pi_j P_{ji} = \pi_i P_{ij} \quad i, j = 0, 1, \ldots \]

- Imply the GBE
- Need not hold for a given Markov chain
- Greatly simplify the calculation of stationary distribution

Methodology:
- Assume DBE hold – have to guess their form
- Solve the system defined by DBE and \( \sum_i \pi_i = 1 \)
  - If system is inconsistent, then DBE do not hold
  - If system has a solution \( \{ \pi_i: i=0, 1, \ldots \} \), then this is the unique stationary distribution
Generalized Markov Chains

- Markov chain on a set of states \( \{0,1,...\} \), that whenever enters state \( i \)
  - The next state that will be entered is \( j \) with probability \( P_{ij} \)
  - Given that the next state entered will be \( j \), the time it spends at state \( i \) until the transition occurs is a RV with distribution \( F_{ij} \)

- \( \{Z(t): t \geq 0\} \) describing the state the chain is in at time \( t \): Generalized Markov chain, or Semi-Markov process

- It does not have the Markov property: future depends on
  - The present state, and
  - The length of time the process has spent in this state
Generalized Markov Chains

- $T_i$: time process spends at state $i$, before making a transition – holding time
- Probability distribution function of $T_i$
  \[
  H_i(t) = P\{T_i \leq t\} = \sum_{j=0}^{\infty} P\{T_i \leq t \mid \text{next state } j\}P_j = \sum_{j=0}^{\infty} F_{ij}(t)P_j
  \]
  \[
  E[T_i] = \int_{0}^{\infty} t \, dH_i(t)
  \]
- $T_{ii}$: time between successive transitions to $i$
- $X_n$ is the $n^{th}$ state visited. \(\{X_n: n=0,1,...\}\)
  - Is a Markov chain: embedded Markov chain
  - Has transition probabilities $P_{ij}$
- Semi-Markov process irreducible: if its embedded Markov chain is irreducible
Limit Theorems

Theorem 3: Irreducible semi-Markov process, \( E[T_{ii}] < \infty \)

- For any state \( j \), the following limit
  \[ p_j = \lim_{t \to \infty} P\{Z(t) = j \mid Z(0) = i\}, \quad i = 0, 1, 2, \ldots \]
  exists and is independent of the initial state.

  \[ p_j = \frac{E[T_j]}{E[T_{jj}]} \]

- \( T_j(t) \): time spent at state \( j \) up to time \( t \)
  \[ P \left\{ p_j = \lim_{t \to \infty} \frac{T_j(t)}{t} \mid Z(0) = i \right\} = 1 \]

- \( p_j \) is equal to the proportion of time spent at state \( j \)
Occupancy Distribution

**Theorem 4:** Irreducible semi-Markov process; $E[T_{ii}] < \infty$. Embedded Markov chain ergodic; stationary distribution $\pi$

\[ \pi_j = \sum_{i=0}^{\infty} \pi_i P_{ij}, \quad j \geq 0; \quad \sum_{i=0}^{\infty} \pi_i = 1 \]

- Occupancy distribution of the semi-Markov process

\[ p_j = \frac{\pi_j E[T_j]}{\sum_i \pi_i E[T_i]}, \quad j = 0, 1, \ldots \]

- $\pi_j$ proportion of transitions into state $j$
- $E[T_j]$ mean time spent at $j$

- Probability of being at $j$ is proportional to $\pi_j E[T_j]$. 
Continuous-Time Markov Chains

Continuous-time process \{X(t): t \geq 0\} taking values in \{0,1,2,...\}. Whenever it enters state i

- Time it spends at state \(i\) is exponentially distributed with parameter \(\nu_i\).
- When it leaves state \(i\), it enters state \(j\) with probability \(P_{ij}\), where \(\sum_{j \neq i} P_{ij} = 1\).

- Continuous-time Markov chain is a semi-Markov process with
  \[ F_{ij}(t) = 1 - e^{-\nu_i t}, \quad i, j = 0,1,... \]

- Exponential holding times: a continuous-time Markov chain has the Markov property.
Continuous-Time Markov Chains

- When at state $i$, the process makes transitions to state $j \neq i$ with rate:
  \[ q_{ij} = \nu_i P_{ij} \]

- Total rate of transitions out of state $i$
  \[ \sum_{j \neq i} q_{ij} = \nu_i \sum_{j \neq i} P_{ij} = \nu_i \]

- Average time spent at state $i$ before making a transition:
  \[ E[T_i] = 1 / \nu_i \]
Occupancy Probability

- Irreducible and regular continuous-time Markov chain
  - Embedded Markov chain is irreducible
  - Number of transitions in a finite time interval is finite with probability 1
- From Theorem 3: for any state $j$, the limit
  \[ p_j = \lim_{t \to \infty} P\{X(t) = j \mid X(0) = i\}, \quad i = 0, 1, 2, \ldots \]
  exists and is independent of the initial state
- $p_j$ is the steady-state occupancy probability of state $j$
- $p_j$ is equal to the proportion of time spent at state $j$ [Why?]
Global Balance Equations

- Two possibilities for the occupancy probabilities:
  - $p_j = 0$, for all $j$
  - $p_j > 0$, for all $j$, and $\sum_j p_j = 1$

- Global Balance Equations
  \[ p_j \sum_{i \neq j} q_{ji} = \sum_{i \neq j} p_i q_{ij}, \quad j = 0, 1, \ldots \]

- Rate of transitions out of $j = \text{rate of transitions into } j$

- If a distribution $\{p_j : j = 0, 1, \ldots\}$ satisfies GBE, then it is the \textit{unique} occupancy distribution of the Markov chain

- Alternative form of GBE:
  \[ \sum_{j \in S} p_j \sum_{i \notin S} q_{ji} = \sum_{i \notin S} p_i \sum_{j \in S} q_{ij}, \quad S \subseteq \{0, 1, \ldots\} \]
Detailed Balance Equations

- Detailed Balance Equations
  \[ p_j q_{ji} = p_i q_{ij}, \quad i, j = 0, 1, ... \]

- Simplify the calculation of the stationary distribution
- Need not hold for any given Markov chain
- Examples: birth-death processes, and reversible Markov chains
Transitions only between neighboring states

\( q_{i,i+1} = \lambda_i, \quad q_{i,i-1} = \mu_i, \quad q_{ij} = 0, \quad |i - j| > 1 \)

Detailed Balance Equations

\[ \lambda_n p_n = \mu_{n+1} p_{n+1}, \quad n = 0, 1, \ldots \]

Proof: GBE with \( S = \{0, 1, \ldots, n\} \) give:

\[ \sum_{j=0}^{n} \sum_{i=n+1}^{\infty} p_j q_{ji} = \sum_{j=0}^{n} \sum_{i=n+1}^{\infty} p_i q_{ij} \Rightarrow \lambda_n p_n = \mu_{n+1} p_{n+1} \]
Birth-Death Process

\[ \mu_n p_n = \lambda_{n-1} p_{n-1} \Rightarrow \]

\[ p_n = \frac{\lambda_{n-1}}{\mu_n} p_{n-1} = \frac{\lambda_{n-1}}{\mu_n} \frac{\lambda_{n-2}}{\mu_{n-1}} p_{n-2} = \ldots = \frac{\lambda_{n-1} \lambda_{n-2} \ldots \lambda_0}{\mu_n \mu_{n-1} \ldots \mu_1} p_0 = p_0 \prod_{i=0}^{n-1} \frac{\lambda_i}{\mu_{i+1}} \]

\[ \sum_{n=0}^{\infty} p_n = 1 \Leftrightarrow p_0 \left[ 1 + \sum_{n=1}^{\infty} \prod_{i=0}^{n-1} \frac{\lambda_i}{\mu_{i+1}} \right] = 1 \Leftrightarrow p_0 = \left[ 1 + \sum_{n=1}^{\infty} \prod_{i=0}^{n-1} \frac{\lambda_i}{\mu_{i+1}} \right]^{-1} \quad \text{if} \quad \sum_{n=1}^{\infty} \prod_{i=0}^{n-1} \frac{\lambda_i}{\mu_{i+1}} < \infty \]

- Use DBE to determine state probabilities as a function of \( p_0 \)
- Use the probability conservation law to find \( p_0 \)

- Using DBE in problems:
  - Prove that DBE hold, or
  - Justify validity (e.g. reversible process), or
  - Assume they hold – have to guess their form – and solve system
**M/M/1 Queue**

- Arrival process: Poisson with rate $\lambda$
- Service times: iid, exponential with parameter $\mu$
- Service times and interarrival times: independent
- Single server
- Infinite waiting room
- $N(t)$: Number of customers in system at time $t$ (state)
M/M/1 Queue

- **Birth-death process → DBE**
  \[ \mu p_n = \lambda p_{n-1} \Rightarrow \]
  \[ p_n = \frac{\lambda}{\mu} p_{n-1} = \rho p_{n-1} = \ldots = \rho^n p_0 \]

- **Normalization constant**
  \[ \sum_{n=0}^{\infty} p_n = 1 \Leftrightarrow p_0 \left[ 1 + \sum_{n=1}^{\infty} \rho^n \right] = 1 \Leftrightarrow p_0 = 1 - \rho \text{, if } \rho < 1 \]

- **Stationary distribution**
  \[ p_n = \rho^n (1 - \rho), \quad n = 0, 1, \ldots \]
The M/M/1 Queue

- **Average number of customers**

\[ N = \sum_{n=0}^{\infty} np_n = (1 - \rho) \sum_{n=0}^{\infty} n\rho^n = (1 - \rho)\rho \sum_{n=0}^{\infty} n\rho^{n-1} \]

\[ \Rightarrow N = \rho(1 - \rho) \frac{1}{(1 - \rho)^2} = \frac{\rho}{1 - \rho} = \frac{\lambda}{\mu - \lambda} \]

- **Applying Little’s Theorem, we have**

\[ T = \frac{N}{\lambda} = \frac{1}{\lambda} \frac{\lambda}{\mu - \lambda} = \frac{1}{\mu - \lambda} \]

- **Similarly, the average waiting time and number of customers in the queue is given by**

\[ W = T - \frac{1}{\mu} = \frac{\rho}{\mu - \lambda} \quad \text{and} \quad N_Q = \lambda W = \frac{\rho^2}{1 - \rho} \]