Topics

- Time-Reversal of Markov Chains
- Reversibility
- Truncating a Reversible Markov Chain
- Burke’s Theorem
- Queues in Tandem
Time-Reversed Markov Chains

- \( \{X_n: n=0,1,\ldots\} \) irreducible aperiodic Markov chain with transition probabilities \( P_{ij} \)
  \[
  \sum_{j=0}^{\infty} P_{ij} = 1, \quad i = 0,1,\ldots
  \]

- Unique stationary distribution (\( \pi_j > 0 \)) if and only if:
  \[
  \pi_j = \sum_{i=0}^{\infty} \pi_i P_{ij}, \quad j = 0,1,\ldots
  \]

- Process in steady state:
  \[
  \Pr\{X_n = j\} = \pi_j = \lim_{n \to \infty} \Pr\{X_n = j \mid X_0 = i\}
  \]
  - Starts at \( n=-\infty \), that is \( \{X_n: n = \ldots,-1,0,1,\ldots\} \)
  - Choose initial state according to the stationary distribution

- How does \( \{X_n\} \) look “reversed” in time?
Time-Reversed Markov Chains

- Define $Y_n = X_{\tau-n}$, for arbitrary $\tau > 0$
- $\{Y_n\}$ is the reversed process.

**Proposition 1:**
- $\{Y_n\}$ is a Markov chain with transition probabilities:
  
  $$P^*_ij = \frac{\pi_j \pi_{ji}}{\pi_i}, \quad i, j = 0, 1, ...$$

- $\{Y_n\}$ has the same stationary distribution $\pi_j$ with the forward chain $\{X_n\}$
Proof of Proposition 1:

\[ P_{ij}^* = P\{Y_m = j \mid Y_{m-1} = i, Y_{m-2} = i_2, \ldots, Y_{m-k} = i_k \} \]
\[ = P\{X_{\tau-m} = j \mid X_{\tau-m+1} = i, X_{\tau-m+2} = i_2, \ldots, X_{\tau-m+k} = i_k \} \]
\[ = P\{X_n = j \mid X_{n+1} = i, X_{n+2} = i_2, \ldots, X_{n+k} = i_k \} \]
\[ = \frac{P\{X_n = j, X_{n+1} = i, X_{n+2} = i_2, \ldots, X_{n+k} = i_k \}}{P\{X_{n+1} = i, X_{n+2} = i_2, \ldots, X_{n+k} = i_k \}} \]
\[ = \frac{P\{X_{n+2} = i_2, \ldots, X_{n+k} = i_k \mid X_n = j, X_{n+1} = i\}P\{X_n = j, X_{n+1} = i\}}{P\{X_{n+2} = i_2, \ldots, X_{n+k} = i_k \mid X_{n+1} = i\}P\{X_{n+1} = i\}} \]
\[ = \frac{P\{X_n = j, X_{n+1} = i\}}{P\{X_{n+1} = i\}} = P\{X_n = j \mid X_{n+1} = i\} = P\{Y_m = j \mid Y_{m-1} = i\} \]
\[ = \frac{P\{X_{n+1} = i \mid X_n = j\}P\{X_n = j\}}{P\{X_{n+1} = i\}} = \frac{P_{ji} \pi_j}{\pi_i} \]
\[ \sum_{i=0}^{\infty} \pi_i P_{ij}^* = \sum_{i=0}^{\infty} \pi_i \frac{\pi_j P_{ji}}{\pi_i} = \pi_j \sum_{i=0}^{\infty} P_{ji} = \pi_j \]
Reversibility

- Stochastic process \( \{X(t)\} \) is called reversible if 
  \( (X(t_1), X(t_2), \ldots, X(t_n)) \) and \( (X(\tau-t_1), X(\tau-t_2), \ldots, X(\tau-t_n)) \) 
  have the same probability distribution, for all \( \tau, t_1, \ldots, t_n \)

- Markov chain \( \{X_n\} \) is reversible if and only if the transition 
  probabilities of forward and reversed chains are equal \( P_{ij} = P_{ij}^* \) 
  or equivalently, if and only if 
  \[
  \pi_i P_{ij} = \pi_j P_{ji}, \quad i, j = 0, 1, \ldots
  \]

  - Detailed Balance Equations \( \leftrightarrow \) Reversibility
Reversibility – Discrete-Time Chains

Theorem 1: If there exists a set of positive numbers \( \{\pi_j\} \), that sum up to 1 and satisfy:

\[
\pi_i P_{ij} = \pi_j P_{ji}, \quad i, j = 0, 1, \ldots
\]

Then:

1. \( \{\pi_j\} \) is the unique stationary distribution
2. The Markov chain is reversible

Example: Discrete-time birth-death processes are reversible, since they satisfy the DBE
Example: Birth-Death Process

- One-dimensional Markov chain with transitions only between neighboring states: \( P_{ij} = 0 \), if \( |i-j| > 1 \)
- Detailed Balance Equations (DBE)
  \[
  \pi_n P_{n,n+1} = \pi_{n+1} P_{n+1,n} \quad n = 0,1,\ldots
  \]
  
- Proof: GBE with \( S = \{0,1,\ldots,n\} \) give:
  \[
  \sum_{j=0}^{n} \sum_{i=n+1}^{\infty} \pi_j P_{ji} = \sum_{j=0}^{n} \sum_{i=n+1}^{\infty} \pi_i P_{ij} \Rightarrow \pi_n P_{n,n+1} = \pi_{n+1} P_{n+1,n}
  \]
Theorem 2: Irreducible Markov chain with transition probabilities $P_{ij}$. If there exist:

- A set of transition probabilities $Q_{ij}$, with $\sum_j Q_{ij} = 1$, $i \geq 0$, and
- A set of positive numbers $\{\pi_j\}$, that sum up to 1, such that

$$\pi_i P_{ij} = \pi_j Q_{ji}, \quad i, j = 0, 1, \ldots$$

(1)

Then:

- $Q_{ij}$ are the transition probabilities of the reversed chain, and
- $\{\pi_j\}$ is the stationary distribution of the forward and the reversed chains

Remark: Use to find the stationary distribution, by guessing the transition probabilities of the reversed chain – even if the process is not reversible
Continuous-Time Markov Chains

- \{X(t): -\infty < t < \infty\} irreducible aperiodic Markov chain with transition rates \( q_{ij}, \ i \neq j \)
- Unique stationary distribution \((p_i > 0)\) if and only if:
  \[
p_j \sum_{i \neq j} q_{ji} = \sum_{i \neq j} p_i q_{ij}, \quad j = 0, 1, ...
\]
- Process in steady state – e.g., started at \( t = -\infty \):
  \[
  \Pr\{X(t) = j\} = p_j = \lim_{t \to -\infty} \Pr\{X(t) = j \mid X(0) = i\}
  \]
- If \( \{\pi_j\} \), is the stationary distribution of the embedded discrete-time chain:
  \[
p_j = \frac{\pi_j}{\nu_j}, \quad \nu_j = \sum_{i \neq j} q_{ji}, \quad j = 0, 1, ...
  \]
Reversed Continuous-Time Markov Chains

- Reversed chain \( \{Y(t)\} \), with \( Y(t) = X(\tau - t) \), for arbitrary \( \tau > 0 \)
- Proposition 2:
  1. \( \{Y(t)\} \) is a continuous-time Markov chain with transition rates:
     \[ q_{ij}^* = \frac{p_j q_{ji}}{p_i}, \quad i, j = 0, 1, \ldots, i \neq j \]
  2. \( \{Y(t)\} \) has the same stationary distribution \( \{p_j\} \) with the forward chain

Remark: The transition rate out of state \( i \) in the reversed chain is equal to the transition rate out of state \( i \) in the forward chain

\[
\sum_{j \neq i} q_{ij}^* = \frac{\sum_{j \neq i} p_j q_{ji}}{p_i} = \frac{p_i \sum_{j \neq i} q_{ij}}{p_i} = \sum_{j \neq i} q_{ij} = \nu_i, \quad i = 0, 1, \ldots
\]
Markov chain \{X(t)\} is reversible if and only if the transition rates of forward and reversed chains are equal \(q_{ij} = q_{ji}\), or equivalently

\[ p_i q_{ij} = p_j q_{ji}, \quad i, j = 0,1,..., i \neq j \]

- Detailed Balance Equations \(\leftrightarrow\) Reversibility

**Theorem 3:** If there exists a set of positive numbers \(\{p_j\}\), that sum up to 1 and satisfy:

\[ p_i q_{ij} = p_j q_{ji}, \quad i, j = 0,1,..., i \neq j \]

Then:
1. \(\{p_j\}\) is the unique stationary distribution
2. The Markov chain is reversible
Example: Birth-Death Process

- Transitions only between neighboring states
  \[ q_{i,i+1} = \lambda_i, \quad q_{i,i-1} = \mu_i, \quad q_{ij} = 0, \quad |i - j| > 1 \]
- Detailed Balance Equations
  \[ \lambda_n p_n = \mu_{n+1} p_{n+1}, \quad n = 0, 1, \ldots \]
- Proof: GBE with \( S = \{0, 1, \ldots, n\} \) give:
  \[ \sum_{j=0}^{n} \sum_{i=n+1}^{\infty} p_j q_{ji} = \sum_{j=0}^{n} \sum_{i=n+1}^{\infty} p_i q_{ij} \Rightarrow \lambda_n p_n = \mu_{n+1} p_{n+1} \]

- M/M/1, M/M/c, M/M/∞
Theorem 4: Irreducible continuous-time Markov chain with transition rates $q_{ij}$. If there exist:
- A set of transition rates $\varphi_{ij}$, with $\sum_{j \neq i} \varphi_{ij} = \sum_{j \neq i} q_{ij}$, $i \geq 0$, and
- A set of positive numbers \{p_j\}, that sum up to 1, such that

$$p_i \varphi_{ij} = p_j q_{ji}, \quad i, j = 0, 1, \ldots, i \neq j$$

Then:
- $\varphi_{ij}$ are the transition rates of the reversed chain, and
- \{p_j\} is the stationary distribution of the forward and the reversed chains.

Remark: Use to find the stationary distribution, by guessing the transition probabilities of the reversed chain – even if the process is not reversible.
Theorem 5:
- For a Markov chain form a graph, where states are the nodes, and for each $q_{ij}>0$, there is a directed arc $i \rightarrow j$
- Irreducible Markov chain, with transition rates that satisfy $q_{ij}>0 \iff q_{ji}>0$
- If graph is a tree – contains no loops – then Markov chain is reversible

Remarks:
- Sufficient condition for reversibility
- Generalization of one-dimensional birth-death process
Kolmogorov’s Criterion (Discrete Chain)

- Detailed balance equations determine whether a Markov chain is reversible or not, based on stationary distribution and transition probabilities.
- Should be able to derive a reversibility criterion based only on the transition probabilities!

**Theorem 6:** A discrete-time Markov chain is reversible if and only if:

\[
P_{i_1i_2} P_{i_2i_3} \cdots P_{i_{n-1}i_n} P_{i_ni_1} = P_{i_1i_n} P_{i_2i_{n-1}} \cdots P_{i_{n-2}i_2} P_{i_{n-1}i_1}
\]

for any finite sequence of states: \( i_1, i_2, \ldots, i_n \), and any \( n \)

- **Intuition:** Probability of traversing any loop \( i_1 \rightarrow i_2 \rightarrow \ldots \rightarrow i_n \rightarrow i_1 \) is equal to the probability of traversing the same loop in the reverse direction \( i_1 \rightarrow i_n \rightarrow \ldots \rightarrow i_2 \rightarrow i_1 \)
Kolmogorov’s Criterion (Continuous Chain)

- Detailed balance equations determine whether a Markov chain is reversible or not, based on stationary distribution and transition rates.
  - Should be able to derive a reversibility criterion based only on the transition rates!

- **Theorem 7:** A continuous-time Markov chain is reversible if and only if:

\[
q_{i_1 i_2} q_{i_2 i_3} \cdots q_{i_{n-1} i_n} q_{i_n i_1} = q_{i_1 i_n} q_{i_n i_{n-1}} \cdots q_{i_3 i_2} q_{i_2 i_1}
\]

for any finite sequence of states: \(i_1, i_2, \ldots, i_n\), and any \(n\)

- **Intuition:** Product of transition rates along any loop \(i_1 \rightarrow i_2 \rightarrow \ldots \rightarrow i_n \rightarrow i_1\) is equal to the product of transition rates along the same loop traversed in the reverse direction \(i_1 \rightarrow i_n \rightarrow \ldots \rightarrow i_2 \rightarrow i_1\)
Kolmogorov’s Criterion (proof)

Proof of Theorem 6:

- **Necessary:** If the chain is reversible the DBE hold

\[
\begin{align*}
\pi_1 P_{i_2} &= \pi_2 P_{i_1} \\
\pi_2 P_{i_3} &= \pi_3 P_{i_2} \\
&\vdots \\
\pi_{n-1} P_{i_{n-1}} &= \pi_n P_{i_{n-1}} \\
\pi_n P_{i_1} &= \pi_1 P_{i_n}
\end{align*}
\]

\[
\Rightarrow P_{i_2} P_{i_3} \cdots P_{i_{n-1}} P_{i_1} = P_{i_1} P_{i_2} \cdots P_{i_{n-1}} P_{i_n}
\]

- **Sufficient:** Fixing two states \(i_1 = i\), and \(i_n = j\) and summing over all states \(i_2, \ldots, i_{n-1}\) we have

\[
P_{i_2} P_{i_3} \cdots P_{i_{n-1}} P_{ji} = P_{ij} P_{ji} \cdots P_{i_2} P_{i_1,j} \Rightarrow P_{ij} P_{ji} = P_{ij} P_{ji}
\]

Taking the limit \(n \to \infty\)

\[
\lim_{n \to \infty} P_{ij} P_{ji} = P_{ij} \cdot \lim_{n \to \infty} P_{ji} \Rightarrow \pi_j P_{ji} = P_{ij} \pi_i
\]
Example: M/M/2 Queue with Heterogeneous Servers

- M/M/2 queue. Servers A and B with service rates $\mu_A$ and $\mu_B$ respectively. When the system empty, arrivals go to A with probability $\alpha$ and to B with probability $1-\alpha$. Otherwise, the head of the queue takes the first free server.

- Need to keep track of which server is busy when there is 1 customer in the system. Denote the two possible states by: 1A and 1B.

- Reversibility: we only need to check the loop $0\rightarrow 1A\rightarrow 2\rightarrow 1B\rightarrow 0$:
  \[ q_{0,1A}q_{1A,2}q_{2,1B}q_{1B,0} = \alpha \lambda \cdot \lambda \cdot \mu_A \cdot \mu_B \quad q_{0,1B}q_{1B,2}q_{2,1A}q_{1A,0} = (1-\alpha) \lambda \cdot \lambda \cdot \mu_B \cdot \mu_A \]

- Reversible if and only if $\alpha=1/2$.

  - What happens when $\mu_A=\mu_B$, and $\alpha\neq1/2$?
Example: M/M/2 Queue with Heterogeneous Servers

\[ p_n = p_2 \left( \frac{\lambda}{\mu_A + \mu_B} \right)^{n-2}, \quad n = 2, 3, \ldots \]

\[
\begin{align*}
\lambda p_0 &= \mu_A p_{1A} + \mu_B p_{1B} \\
(\mu_A + \mu_B) p_2 &= \lambda (p_{1A} + p_{1B}) \\
(\mu_A + \lambda) p_{1A} &= \alpha \lambda p_0 + \mu_B p_2
\end{align*}
\]

\[
\begin{align*}
p_{1A} &= p_0 \frac{\lambda}{\mu_A} \frac{\lambda + \alpha (\mu_A + \mu_B)}{2 \lambda + \mu_A + \mu_B} \\
p_{1B} &= p_0 \frac{\lambda}{\mu_B} \frac{\lambda + (1 - \alpha) (\mu_A + \mu_B)}{2 \lambda + \mu_A + \mu_B} \\
p_2 &= p_0 \frac{\lambda^2}{\mu_A \mu_B} \frac{\lambda + (1 - \alpha) \mu_A + \alpha \mu_B}{2 \lambda + \mu_A + \mu_B}
\end{align*}
\]

\[ p_0 + p_{1A} + p_{1B} + \sum_{n=2}^{\infty} p_n = 1 \implies p_0 = \left[ 1 + \frac{\lambda}{\mu_A + \mu_B - \lambda} \frac{\lambda^2}{\mu_A \mu_B} \frac{\lambda + (1 - \alpha) \mu_A + \alpha \mu_B}{2 \lambda + \mu_A + \mu_B} \right]^{-1} \]
Multidimensional Markov Chains

Theorem 8:
- \{X_1(t)\}, \{X_2(t)\}: independent Markov chains
- \{X_i(t)\}: reversible
- \{X(t)\}, with \(X(t)=(X_1(t), X_2(t))\): vector-valued stochastic process
  - \{X(t)\} is a Markov chain
  - \{X(t)\} is reversible

Multidimensional Chains:
- Queueing system with two classes of customers, each having its own stochastic properties – track the number of customers from each class
- Study the “joint” evolution of two queueing systems – track the number of customers in each system
Example: Two Independent M/M/1 Queues

- Two independent M/M/1 queues. The arrival and service rates at queue $i$ are $\lambda_i$ and $\mu_i$ respectively. Assume $\rho_i = \lambda_i/\mu_i < 1$.
- $\{(N_1(t), N_2(t))\}$ is a Markov chain.
- Probability of $n_1$ customers at queue 1, and $n_2$ at queue 2, at steady-state
  \[ p(n_1, n_2) = (1 - \rho_1)\rho_1^{n_1} \cdot (1 - \rho_2)\rho_2^{n_2} = p_1(n_1) \cdot p_2(n_2) \]
- “Product-form” distribution
- Generalizes for any number $K$ of independent queues, M/M/1, M/M/$c$, or M/M/$\infty$. If $p_i(n_i)$ is the stationary distribution of queue $i$:
  \[ p(n_1, n_2, \ldots, n_K) = p_1(n_1)p_2(n_2)\ldots p_K(n_K) \]
Example: Two Independent M/M/1 Queues

- Stationary distribution:
  \[ p(n_1, n_2) = \left(1 - \frac{\lambda_1}{\mu_1}\right) \left(1 - \frac{\lambda_2}{\mu_2}\right) p_0 \]

- Detailed Balance Equations:
  \[ \mu_1 p(n_1 + 1, n_2) = \lambda_1 p(n_1, n_2) \]
  \[ \mu_2 p(n_1, n_2 + 1) = \lambda_2 p(n_1, n_2) \]

- Verify that the Markov chain is reversible – Kolmogorov criterion
Truncation of a Reversible Markov Chain

- **Theorem 9**: \{X(t)\} reversible Markov process with state space \(S\), and stationary distribution \(\{p_j: j \in S\}\). Truncated to a set \(E \subset S\), such that the resulting chain \(\{Y(t)\}\) is irreducible. Then, \(\{Y(t)\}\) is reversible and has stationary distribution:

\[
\tilde{p}_j = \frac{p_j}{\sum_{k \in E} p_k}, \quad j \in E
\]

- **Remark**: This is the conditional probability that, in steady-state, the original process is at state \(j\), given that it is somewhere in \(E\)

- **Proof**: Verify that:

\[
\tilde{p}_j q_{ji} = \tilde{p}_i q_{ij} \iff \frac{p_j}{\sum_{k \in E} p_k} q_{ji} = \frac{p_i}{\sum_{k \in E} p_k} q_{ij} \iff p_j q_{ji} = p_i q_{ij}, \quad i, j \in S; i \neq j
\]

\[
\sum_{j \in E} \tilde{p}_j = \sum_{j \in E} \frac{p_j}{\sum_{k \in E} p_k} = 1
\]
Example: Two Queues with Joint Buffer

- The two independent M/M/1 queues of the previous example share a common buffer of size $B$ — arrival that finds $B$ customers *waiting* is blocked
- State space restricted to $E = \{(n_1, n_2): (n_1 - 1)^+ + (n_2 - 1)^+ \leq B\}$
- Distribution of truncated chain: $p(n_1, n_2) = p(0,0) \cdot \rho_1^{n_1} \rho_2^{n_2}$, $(n_1, n_2) \in E$
- Normalizing:
  $$p(0,0) = \left[ \sum_{(n_1, n_2) \in E} \rho_1^{n_1} \rho_2^{n_2} \right]^{-1}$$

- Theorem specifies joint distribution up to the normalization constant
- Calculation of normalization constant is often tedious

State diagram for $B = 2$
Burke’s Theorem

- \{X(t)\} birth-death process with stationary distribution \{p_j\}
- Arrival epochs: points of increase for \{X(t)\}
  Departure epoch: points of increase for \{X(t)\}
- \{X(t)\} completely determines the corresponding arrival and departure processes

Diagram:
- Arrows indicating arrivals and departures over time.
Burke’s Theorem

- Poisson arrival process: $\lambda_j = \lambda$, for all $j$
  - Birth-death process called a $(\lambda, \mu_j)$-process
  - Examples: M/M/1, M/M/c, M/M/$\infty$ queues
- Poisson arrivals $\rightarrow$ LAA:
  For any time $t$, future arrivals are independent of $\{X(s): s \leq t\}$
- $(\lambda, \mu_j)$-process at steady state is reversible: forward and reversed chains are stochastically identical
  - Arrival processes of the forward and reversed chains are stochastically identical
- Arrival process of the reversed chain is Poisson with rate $\lambda$
- The arrival epochs of the reversed chain are the departure epochs of the forward chain
  - Departure process of the forward chain is Poisson with rate $\lambda$
Burke’s Theorem

- Reversed chain: arrivals after time $t$ are independent of the chain history up to time $t$ (LAA)
- Forward chain: departures prior to time $t$ and future of the chain $\{X(s): s \geq t\}$ are independent
Burke’s Theorem

Theorem 10: Consider an M/M/1, M/M/c, or M/M/∞ system with arrival rate \( \lambda \). Suppose that the system starts at steady-state. Then:

1. The departure process is Poisson with rate \( \lambda \)
2. At each time \( t \), the number of customers in the system is independent of the departure times prior to \( t \)

Fundamental result for study of networks of M/M/* queues, where output process from one queue is the input process of another
Customers arrive at queue 1 according to Poisson process with rate $\lambda$.

- Service times exponential with mean $1/\mu_i$. Assume service times of a customer in the two queues are independent.
- Assume $\rho_i = \lambda/\mu_i < 1$

What is the joint \textit{stationary} distribution of $N_1$ and $N_2$ – number of customers in each queue?

- Result: in \textit{steady state} the queues are independent and

$$p(n_1, n_2) = (1 - \rho_1)\rho_1^{n_1} \cdot (1 - \rho_2)\rho_2^{n_2} = p_1(n_1) \cdot p_2(n_2)$$
Single-Server Queues in Tandem

- Q1 is a M/M/1 queue. At steady state its departure process is Poisson with rate $\lambda$. Thus Q2 is also M/M/1.

- Marginal stationary distributions:
  
  \[
  p_1(n_1) = (1 - \rho_1) \rho_1^{n_1}, \quad n_1 = 0, 1, \ldots
  \]
  
  \[
  p_2(n_2) = (1 - \rho_2) \rho_2^{n_2}, \quad n_2 = 0, 1, \ldots
  \]

- To complete the proof: establish independence at steady state

- Q1 at steady state: at time $t$, $N_1(t)$ is independent of departures prior to $t$, which are arrivals at Q2 up to $t$. Thus $N_1(t)$ and $N_2(t)$ independent:

  \[
  P\{N_1(t) = n_1, N_2(t) = n_2\} = P\{N_1(t) = n_1\} P\{N_2(t) = n_2\} = p_1(n_1) \cdot P\{N_2(t) = n_2\}
  \]

- Letting $t \to \infty$, the joint stationary distribution

  \[
  p(n_1, n_2) = p_1(n_1) \cdot p_2(n_2) = (1 - \rho_1) \rho_1^{n_1} \cdot (1 - \rho_2) \rho_2^{n_2}
  \]
Queues in Tandem

- **Theorem 11**: Network consisting of $K$ single-server queues in tandem. Service times at queue $i$ exponential with rate $\mu_i$, independent of service times at any queue $j \neq i$. Arrivals at the first queue are Poisson with rate $\lambda$. The stationary distribution of the network is:

$$p(n_1, \ldots, n_K) = \prod_{i=1}^{K} (1 - \rho_i) \rho_i^{n_i}, \quad n_i = 0, 1, \ldots; i = 1, \ldots, K$$

- At steady state the queues are independent; the distribution of queue $i$ is that of an isolated M/M/1 queue with arrival and service rates $\lambda$ and $\mu_i$

$$p_i(n_i) = (1 - \rho_i) \rho_i^{n_i}, \quad n_i = 0, 1, \ldots$$

- Are the queues independent if not in steady state? Are stochastic processes $\{N_1(t)\}$ and $\{N_2(t)\}$ independent?
Theorem 12: Network consisting of $K$ queues in tandem. Service times at queue $i$ exponential with rate $\mu_i(n_i)$ when there are $n_i$ customers in the queue – independent of service times at any queue $j\neq i$. Arrivals at the first queue are Poisson with rate $\lambda$. The stationary distribution of the network is:

$$p(n_1,\ldots,n_K) = \prod_{i=1}^{K} p_i(n_i), \quad n_i = 0,1,\ldots; i = 1,\ldots,K$$

where \(\{p_i(n_i)\}\) is the stationary distribution of queue $i$ in isolation with Poisson arrivals with rate $\lambda$.

Examples: \(/\text{M}/c\) and \(/\text{M}/\infty\) queues

- If queue $i$ is \(/\text{M}/\infty\), then:

$$p_i(n_i) = \frac{(\lambda / \mu_i)^n_i}{n_i!} e^{-\lambda / \mu_i}, \quad n_i = 0,1,...$$