Positive semi-definite cone

\[ S^n = \{ X \in \mathbb{R}^{n \times n} \mid X = X^T \} \] symmetric matrices

\[ S_n^+ = \{ X \in S^n \mid X \succeq 0 \} \] p.s.d. matrices

\[ S_{++}^n = \{ X \in S^n \mid X \succ 0 \} \] p.o. matrices

If \( A, B \in S_n^+ \) then \( \theta, A + \theta \cdot B \in S_n^+ \), \( \theta_i, \theta_2 \)

Operations that preserve convexity

1. Intersections

   If \( S_1, S_2 \) convex \( \Rightarrow S_1 \cap S_2 \) is convex

   Extends to infinite number of sets.

   Let \( S_x \) be convex \( \quad \forall x \in A \)

   \( \Rightarrow \quad \bigcap \_{x \in A} S_x \) is convex

   \[ \text{Q.1.} \quad S_n^+ = \bigcap \{ X \in S^n \mid Z^T X Z \geq 0 \} \]

   \[ Z \neq 0 \]

   \( \rightarrow \) halfspaces in \( S^n \)

In fact, every closed convex set can be represented as an intersection of (infinite) halfspaces.
2. **Affine Transform**

\[ f : \mathbb{R}^n \rightarrow \mathbb{R}^m \text{ is affine if} \]

\[ f(x) = Ax + b \quad \text{for some } A \in \mathbb{R}^{m \times n}, b \in \mathbb{R}^m \]

(linear if \( b = 0 \))

- If \( S \) is convex and \( f \) is affine, then image of \( f(S) = \{ f(x) \mid x \in S \} \) is convex.

Also, inverse image

\[ f^{-1}(S) = \{ x \mid f(x) \in S \} \text{ is convex} \]

**Example:**

1. **Scaling:** \( \alpha S = \{ \alpha x \mid x \in S \} \) is convex.
2. **Translation:** \( S + \alpha = \{ x + \alpha \mid x \in S \} \) is convex.

3. **Projection:** If \( S \subseteq \mathbb{R}^m \times \mathbb{R}^n \)

\[ T = \{ x_1 \in \mathbb{R}^m \mid (x_1, x_2) \in S \text{ for some } x_2 \in \mathbb{R}^n \} \]

4. **Sum:**

\[ S_1 + S_2 = \{ x + y \mid x \in S_1, y \in S_2 \} \text{ is convex} \]

**Proof:**

\[ S_1 \times S_2 = \{ (x_1, x_2) \mid x_1 \in S_1, x_2 \in S_2 \} \]

in convex

\[ f(x_1, x_2) = x_1 + x_2 \]

5. **Partial sum:**

\[ S = \{ (x_1, x_2, x_3) \mid (x_1, x_2) \in S_1, (x_1, x_3) \in S_2 \} \]

**Proof:**

\[ P_1 = \{ x_1 \in \mathbb{R}^m \mid (x_1, y_1) \in S_1 \} \text{ is convex, similarly } P_2, P_3, P_4 \]

\[ P_1 \cap P_2 \text{ is convex, } P_2 + P_3 \text{ is convex, } (P_1 \cap P_2) \times P_3 \text{ in convex} \]
6. Polyhedron: \( P = \{ x \mid A x \leq b, c^T x = d \} \)
   \( = \{ x \mid f(x) \in \mathbb{R}^n \times \{ 0 \} \} \) is convex
   where \( f(x) = (b - A x, d - c^T x) \) is the outmost

7. Solution set of linear matrix inequality:
   \[ A(x) = x_1 A_1 + \ldots + x_n A_n \leq B \]
   \( \{ x \mid A(x) \leq B \} \) is convex
   \( = \{ x \mid f(x) \geq 0 \} \) is p.s.d. cone
   where \( f(x) = B - A(x) \)

8. Hyperbolic cone:
   \[ H = \{ x \mid x^T P x \leq (c^T x)^2, c^T x > 0 \} \], \( P \in \mathbb{S}^n \)
   \( f(x) = (P^{1/2} x, c^T x) \)
   \( f(x) \geq (f_1(x))^2, f_2(x) > 0 \)
   \( f_1(x) = z, f_2(x) = t \)
   \( H = \{ x \mid z^T z \leq t^2, z > 0, P^{1/2} x = z, c^T x = t \} \)
   2nd order cone.

9. Ellipsoid:
   \( E = \{ x \mid (x - x_c)^T P^{-1} (x - x_c) \leq 1 \} \), \( P \in \mathbb{S}^n \)
   \( f(x) = P^{-1/2} (x - x_c) \)
   \( E = \{ x \mid f(x)^T f(x) \leq 1 \} \)
   \( \uparrow \) circle
Perspective functions - preserve convexity

\[ P : \mathbb{R}^{n+1} \to \mathbb{R}^n \]
\[ \text{dom } P = \mathbb{R}^n \times \mathbb{R}^+ \]
\[ P(z,t) = z/t , \quad z \in \mathbb{R}^n , \quad t \in \mathbb{R}^+ \]

3-d object

Pinhole

Image screen

2-d representation

\[ (x_1/x_3, x_2/x_3, 1) \]

if \( C \subseteq \text{dom } P \) is convex

\[ P^{-1}(C) = \{ P(x) \mid x \in C \} \]

is convex

Inverse image is also convex i.e. if \( C \subseteq \mathbb{R}^n \) is convex then \( P^{-1}(C) = \{ (x,t) \in \mathbb{R}^{n+1} \mid x/t \in C , \ t > 0 \} \)

Linear fractional functions:

Suppose \( g : \mathbb{R}^n \to \mathbb{R}^{m+1} \) is affine i.e.

\[ g(x) = \begin{bmatrix} A & | & b \\ c^T & | & d \end{bmatrix} x \]

Then the function \( f : \mathbb{R}^n \to \mathbb{R}^m \)

\[ f(x) = (Ax+b)/(C^Tx+d) \]

\( \text{dom } f = \{ x \mid C^Tx+d > 0 \} \)

is called linear-fractional
linear-fractionals preserve convexity

i.e. if $C \subseteq \mathbb{R}^n$ is convex then $f(C)$ is convex
if $C \subseteq \mathbb{R}^n$ is convex then $f^{-1}(C)$ is convex

e.g.: conditional probabilities

if $u$ & $v$ r.v. in $\{1, \ldots, n\}$

\[ p_{ij} = \Pr(u = i, v = j) \]

let \[ f_{ij} = \Pr(u = i | v = j) \]

linear-fractional mapping

\[ \lambda = \frac{p_{ij}}{\sum_{k=1}^{n} p_{kj}} \]

if set of probabilities is convex, then set of conditional probabilities also convex
Some questions

2.10
2.12
2.17

Separating & Supporting Hyperplanes

Theorem: Let C & D be convex, \[ C \cap D = \emptyset \]

Then \( \exists a \neq 0, b \) such that

\[ a^T x \leq b \quad \forall x \in C \quad \& \quad a^T x \geq b \quad \forall x \in D \]

Proof: (Special case)

Assume \( \text{dist}(C, D) = \inf \{ \| u - v \|_2 | u \in C, v \in D \} \)

\[ > 0 \]

\( \exists c \in C, d \in D, s.t. \)

\[ \| c - d \|_2 = \text{dist}(C, D) \]

Define \( a = d - c \), \( b = \frac{\| d \|_2^2 - \| c \|_2^2}{2} \)

Then \( f(x) = a^T x - b = (d - c)^T (x - \frac{1}{2}(d + c)) \)

To show \( f(x) > 0 \) if \( x \in D \), \( f(x) \leq 0 \) if \( x \in C \)

Suppose \( \exists u \) s.t.

\[ f(u) = (d - c)^T (u - \frac{1}{2}(d + c)) \leq 0 \]
\[ f(u) = (d-c)^T (u - d + \frac{1}{2} (d-c)) \]
\[ = (d-c)^T (u-d) + \frac{1}{2} \| d-c \|_2^2 \]
\[ f(u) < 0 \implies (d-c)^T (u-d) < 0 \]
\[ \frac{d}{dt} \| d + t(u-d) - c \|_2^2 \big|_{t=0} = 2 (d-c)^T (u-d) < 0 \]

for small \( t > 0 \) with \( t \leq 1 \),
\[ \| d + t(u-d) - c \|_2 < \| d - c \|_2 \]
i.e. \((1-t)d + tu\) is closer to \( c \) than \( d \).

\[ \text{Strict separation} \]

May not hold in general (even when the sets are closed.) Examples where strict separation holds: (Exercise 2.23)

1. A point \( d \) and a closed convex set \( C \) such that \( B(x_0, \epsilon) \cap C = \emptyset \)

Consider the set \( B(x_0, \epsilon) \)
\[ a \perp b \text{ s.t. } a^T x \leq b \quad \forall x \in C \]
\[ \text{such that } a^T x > b \text{ and } \forall x \in B(x_0, \epsilon) \]
\[ B(x_0, \epsilon) = \{ x_0 + u \mid \| u \|_2 \leq \epsilon \} \]
\[ a^T (x_0 + u) \geq b \quad \forall \| u \| \leq \epsilon \]

Let \( u = -\epsilon \frac{a}{\|a\|_2} \)
\[ a^T x_0 - \epsilon \| a \|_2 / 2 \]
\[ f(x) = a^T x - b - \epsilon \| a \|_2 / 2 \]

is negative on \( C \) and positive on \( x_0 \).
Converse: the existence of hyperplane implies C & D are disjoint may not hold in general.

Counter-ex.: C = D = f 0 \in R

Hyperplane \( x = 0 \) separates C & D

Positive results:
1. if C is open & f affine function f
   
   that is non-positive on C & non-negative on D. The C & D are disjoint

Supporting Hyperplanes:

\[ C \subset R^n, \quad x_0 \in \text{bd } C = \text{bd } C \setminus \text{int } C \]

if \( a \neq 0 \) is s.t. \( a^T x < a^T x_0 \) for \( x \in C \)

\( a^T x = a^T x_0 \) is called supporting hyperplane

Then for any non-empty convex set C

\[ x_0 \in \text{bd } C, \quad f \text{ a supporting hyperplane at } x_0 \]

Proof:

If \( \text{int } C \) is non-empty

then apply Separation Hyperplane Theorem to \( \text{int } C \) & \( \{ x_0 \} \)

1. \( \emptyset \neq C \) lies in affine set of dim \( \leq n \)

2. any hyperplane containing the affine set contains C & \( x_0 \), in which case it is a supporting hyperplane.