Elements of Extraterrestrial Thermodynamics: Dark Matter, Dark Energy, and Black Holes

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Abstract

Courses in chemical engineering thermodynamics develop relations and solve problems about phenomena on the Earth. The thermodynamic laws are considered immutable and emphasis is on relating properties to measured variables. The thermodynamics of the cosmos is significantly different due to phenomenological complexities discovered by cosmologists. This paper introduces students and teachers of chemical engineering to the phenomena and thermodynamic treatment of extraterrestrial systems, including the standard model for the developing universe and the properties of black holes.

1 Introduction

Courses and textbooks of chemical engineering thermodynamics treat earthbound, classical Newtonian principles. These are appropriate to the functions and studies of our discipline. Occasionally, however, we teachers of thermodynamics may be asked by inquisitive students about the application of the Laws to the cosmos. Typically, the reply is that we simply don't know enough about outer space phenomena and the principles of general relativity that govern their behavior to give an answer. For example, we know how to use control volumes in Euclidian 3-space, but not in spacetime where time and space form a 4-dimensional continuum. Except near fluid critical points, we can ignore the effects of gravity, but gravity dominates the behavior of the cosmos. Cosmologists have known since the 1920s that the first law about conservation of energy does not apply to intergalactic spacetime[1]. But if energy is not conserved, what thermodynamic equations do apply? How is the second law manifested in the universe?

The literature about the universe is large and complicated, and uncertainties still exist about observed and inferred phenomena. Yet, the fields of astrophysics and quantum field theory have advanced sufficiently to give some authoritative answers to questions about extraterrestrial thermodynamics that chemical engineers might be curious about.

In this paper we introduce the subject of extraterrestrial thermodynamics by highlighting its difference from classical treatments. In the next section, we discuss what we understand about the range and dynamics of the universe as well as energy and entropy. This is followed by a brief development of the standard model of cosmology, which essentially accounts for the known phenomena and properties of intergalactic regions where energy is not conserved. Finally, we discuss the thermodynamics of black holes, for which energy is conserved but the equations for black holes are unfamiliar because of the singularity in spacetime located at their centers.

2 What do we understand about the universe?

Humans have always tried to understand the dynamics of the universe by building devices for measuring and creating theories to explain the phenomena they see. Our current views of the cosmos are based on recent astronomical measurements and theories such as Einstein's general relativity. These are being continually updated from their limited scope and completeness as new data and thinking appear. A concise summary of recent and anticipated advances in theory and observation technology is given in Ref. [2].

2.1 The universe is expanding

The Milky Way galaxy is a rotating disc about 100,000 light-years in diameter. Our sun is located about 25,000 light years from the center of the galaxy and the period of its orbit in the galaxy is about 125,000 years. Our Milky Way galaxy is one of about 100 billion galaxies we can identify using modern telescopes such as the NASA Hubble space telescope. The universe we know arose from the "big bang" 13.8 billion years ago as determined from temperature anisotropies[3] in the cosmic microwave background (CMB). Whatever was going on before then is unknowable because our current spacetime did not exist then. The big bang occurred everywhere at once and the universe has been expanding ever since, at first at a decelerating rate and recently at an accelerating rate (see Figure 1). The accelerating expansion of the universe was discovered by Perlmutter, Schmidt, and Riess using distant supernovae as standard candles (by comparing their observed brightness with their known luminosity), for which they were awarded the 2011 Nobel prize in physics.

The current rate of expansion is given by the Hubble constant $H_\circ$ in the equation

$$v = H_\circ d\,, \tag{2.1}$$

where $H_\circ = 67.4$ km s^{-1} Mpc^{-1} based on results[3] from the final full mission Planck measurements of the CMB anisotropies. v is speed in km s^{-1} and d is distance in Mpc (Mpc stands for 10^6 parsec). This means that a galaxy located at a distance d of 10^8 parsec (1 parsec = 3.26 light-years) would be moving away from us at a velocity of 6740 km s^{-1}, which is about 2.2 percent of the speed of light. This expansion rate applies to any pair of galaxies, not just to us. Of course, the galaxies must be separated by a sufficient distance not to be perturbed by local interactions with each other. For example, the Andromeda galaxy, our nearest comparably-sized neighbor at a distance of 2.5 million light-years, is actually moving toward the Milky Way due to local gravitational forces. These two galaxies are expected to collide in about 4.5 billion years. But at larger distances of the order of

100 million light years, the Hubble expansion rate applies to today's universe for any pair of galaxies.

The oldest galaxy we can observe[4] is identified as GN-z11, which has a redshift of 11.1 (see Eq. (3.17) for definition of redshift). The light observed from this galaxy was emitted when the age of the universe was 400 million years. This galaxy is currently located at a "proper" distance of 32 billion light years from us. This seems incongruous relative to the age of the universe (13.8 billion years) because, according to Hubble's constant, it means that the galaxy is currently moving away from us at a speed of $2.2c$, or more than twice the speed of light. According to special relativity, nothing can travel faster than light. However, we are looking back in time about 13.4 billion years to observe a photon emitted by the galaxy. That photon traversed the distance between the galaxy and the Milky Way at the speed of light while, during its trajectory, the distance increased by $32.0-13.4 = 18.6$ billion light-years. The speed limit of c applies locally for objects in space which is not expanding over the time of the observation.

In the case of GN-z11, we observed an object 32 billion light years away. If this is the most distant galaxy we can see, then the diameter of the observable universe is $2 \times 32 = 64$ billion light years. We can also "see" the CMB radiation formed when the age of the universe was 380,000 years, which extends the diameter of the observable universe to 96 billion light years.

2.2 Tension in value of Hubble constant

A space telescope can be used to derive the Hubble constant directly from (2.1) by measuring the distance d and the velocity v. The distance d of a star is measured by comparing calculated luminosity values with the stars' apparent brightness as seen from Earth. The calculated luminosity values are determined from *standard candles*, which are cepheid variable stars and type Ia supernovae. The velocity v is determined from redshift spectroscopy data. There is a tension between the Planck value of $H_\circ = 67.4$ km s^{-1} Mpc^{-1} derived from CMB anisotropies and the Hubble space telescope value of 74.0 km s^{-1} Mpc^{-1}, a difference of 9%. New physics[5] may be needed to explain the 9% disparity. The likelihood that the difference is a fluke is just 1 in 100,000.

2.3 Natural system of units used by cosmologists

Energy (E) is connected to mass (M) by Einstein's equation

$$E = Mc^2 \,, \tag{2.2}$$

where the speed of light $c = 2.99792458 \times 10^8$ m s^{-1}. In our familiar SI system, energy is measured in J and mass in kg. In the *natural* system of units used by cosmologists, certain fundamental constants (the speed of light, c, the reduced Planck constant, $\hbar$, the vacuum permittivity or so-called electric constant, $\epsilon_\circ$, and the Boltzmann constant, k_B) are set equal to unity and therefore do not appear in equations. This simplifies the equations but can lead to problems in calculations if these natural units are misunderstood. Any variable with SI units of (kg$^\alpha$ m$^\beta$ s$^\gamma$) may be expressed in SI units as

$$(E)^{\alpha-\beta-\gamma}\, \hbar^{\beta+\gamma}\, c^{\beta-2\alpha} \,, \tag{2.3}$$

where E is an arbitrarily chosen energy unit. A popular choice is GeV (1 GeV = 1.6022×10^{-10} J). Setting the constants $\hbar$ and c to unity gives natural units of GeV$^{\alpha-\beta-\gamma}$. Given natural units of $\text{GeV}^{\alpha-\beta-\gamma}$, the desired SI unit can always be recovered by multiplying by the conversion factor $\hbar^{\beta+\gamma}\, c^{\beta-2\alpha}$.

For example, energy density has SI units of (J m^{-3}) = (kg m^{-1} s^{-2}) so $\alpha = 1$, $\beta = -1$, $\gamma = -2$ and the natural unit of energy density is GeV^4. Given the natural unit is GeV^4, the SI unit is obtained by multiplying by the conversion factor $(\hbar c)^{-3}$ to obtain the SI units of J m^{-3}. Time ($\alpha = \beta = 0, \gamma = 1$) in GeV^{-1} is multiplied by a factor of $\hbar$ to obtain SI units of s. Mass ($\alpha = 1, \beta = \gamma = 0$) in GeV is multiplied by a factor of c^{-2} to obtain SI units of kg, as shown already by Eq. (2.2). In the natural system of units, $c = 1$ so (2.2) is reduced to $E = M$. Since energy and mass have the same natural values, both the symbols and words are often used interchangeably. In this paper, SI units are used throughout unless otherwise noted.

2.4 Energy currently consists of ordinary matter, dark matter, and dark energy

We have no experimental evidence that the universe is finite or has a boundary, so it may be infinite. The visible universe is isotropic and homogeneous with no preferred direction or special location, at least on the scale of galactic clusters. Measurements of the CMB radiation [3] show the distribution of energy of the universe. Galaxies are mostly made up of something called *dark matter*, which is 26.6% of the total energy of the universe. Ordinary matter, which includes all matter known to the standard model of particle physics (stars, black holes, dust, photons and neutrinos) makes up only 4.9% of the total energy. The rest of the energy, called *dark energy*, is $100 - 26.6 - 4.9 = 68.5\%$ of the total energy. It is emphasized that the percentages are only for now and change over time. The observable universe began as an infinitely dense and infinitesimally small amount of radiation. As the universe grew, the radiation was replaced mostly by matter, and now the matter is being replaced by dark energy. The radiation energy of photons has faded to less than 0.1% of the total energy. In the distant future, almost all of the energy will be dark energy.

2.5 Dark matter

Even though dark matter is invisible, its presence is clear from the powerful gravitational force it exerts on the regular matter. The mass of a galaxy is about 15% ordinary matter, of which about one-half is dust. The remaining 85% of the mass is so-called dark matter. It is hypothesized that this gravitational interaction is caused by weakly interacting massive particles (wimps), which has not yet been discovered even though it is known that galaxies are surrounded by massive spherical globs of dark matter. When galaxies collide, both the stars and the dark matter pass through each other with little effect, but the dust inside the colliding galaxies generates shock waves which remain in place, emitting radiation from the location of the collision.

2.6 Dark energy

Galaxies are stable collections of orbiting stars. While the galaxies themselves are not expanding, space between them is being created at an accelerating rate. This is surprising because one would expect gravity to pull the galaxies back together. Instead, space is being created by the generation of *dark energy* at a fixed energy density. As a result, energy is not conserved within the universe as a whole. However, atoms are not expanding; our planetary system is not expanding; and our galaxy is not expanding. These structures are "frozen out" of the expansion of the universe by gravitation. Dark energy is said to exert a negative pressure that makes far distant galaxies move apart. On the basis of the First Law of thermodynamics for an adiabatic, closed system

$$dE = -pdV \,. \tag{2.4}$$

On the earth, energy decreases with volume, so p is positive. For dark energy, energy increases with volume, so the pressure p of dark energy is negative.

A theory of dark energy will have to await a theory of quantum gravity. So far, combining the theory of quantum mechanics with Einstein's theory of general relativity has been elusive. In the meantime, dark energy can be thought of as the zero-point energy of a vacuum.

2.7 Entropy and the arrow of time in the cosmos

The second law of thermodynamics is that the entropy of an isolated system always increases. This law applies to the observable universe from the big bang until now. Entropy was extremely low at the time of the creation of the universe because its size was infinitesimally small in comparison to its present diameter of about 96 billion light years. The entropy of the universe has increased continuously from the big bang 13.8 billion years ago to the present day, thus identifying the direction of the arrow of time. The supermassive black holes located at the centers of galaxies are compact trash containers for the huge quantities of entropy generated by the formation and expansion of the universe.

Humans have a sense that there is a direction from the past towards the future. We possess instruments which mark the passage of time, ranging from our bodily pulse rate to digital clocks. We are familiar with the flow of time from past to present towards the future, and think that a video running backwards is funny or unnatural.

Why is the entropy increasing in the direction of the future? Because one billion years ago the entropy was much less, and this reasoning goes all the way back to the big bang when the entropy of the observable universe was close to zero. Because the rate of expansion of the universe is accelerating, its entropy will continue to increase until we are the only galaxy in the sky. The black hole at the center of our galaxy will consume all the burned-out stars. Then, over a period of the order of 10^{100} billion years, the black holes themselves evaporate. This predicts the fate of our universe but offers no explanation for the zero value of entropy at the time of the big bang.

3 The Standard cosmological model

The standard cosmological model is based on the field equations of Einstein, solutions of which are used to model the dynamics of the formation of the universe from about one second after its formation by the big bang until today. This section gives elements of the Einstein field equations and a quantitative solution, the Friedmann equation, which describes the types of energy (radiation, dark energy, dark matter, and ordinary matter) and their evolution with respect to time.

3.1 Einstein's field equations

The Einstein field equations are a set of 10 nonlinear second-order partial differential equations:

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = \frac{8\pi G}{c^4} T_{\mu\nu} \,, \tag{3.1}$$

where $T_{\mu\nu}$ is the energy-momentum tensor in J m^{-3}, $R_{\mu\nu}$ is the Ricci tensor in m^{-2}, R is the Ricci scalar in m^{-2}, $g_{\mu\nu}$ is the dimensionless spacetime metric tensor, and G is Newton's gravitational constant (see Nomenclature). $T_{\mu\nu}$, $R_{\mu\nu}$, and $g_{\mu\nu}$ are 4×4 symmetric tensors.

The left-hand-side of (3.1) describes events in spacetime (t, x, y, z) while the right-hand-side calculates the energy and momentum flux of matter at these events. The solutions for specific problems are complex and difficult to find. Nevertheless, considering that the left-hand side of the equation consists of algebraic operations on the spacetime, and that the right-hand side of the equation is the effect of the matter, we can say: "Spacetime tells matter how to move; matter tells spacetime how to curve"[6].

At one time, Einstein included a term, $\Lambda g_{\mu\nu}$, on the left-hand-side of Eq.(3.1) to stabilize what was considered a static universe. Later, when it became clear that the universe was not static, Einstein abandoned the term. Today, a century later, the cosmological constant is accepted as the explanation for the accelerating expansion of the universe. In this case, physicists prefer to consider the term not to be a part of the left-hand side of Eq. (3.1) (the part that describes the curvature of spacetime), but rather to be an additional term on the right-hand side. However, transferring Λ to the right-hand side of the equation requires dividing by the factor $8\pi G/c^4$ so that the cosmological constant has units of energy density ϵ_Λ in J m^{-3}

$$\epsilon_\Lambda = \frac{\Lambda c^4}{8\pi G} , \tag{3.2}$$

like the other forms of energy for radiation and matter.

3.2 Friedmann equation

One solution to (3.1), based on the assumption of homogeneity and isotropy in the universe over large scales of the order of 100 million light years, is the Friedmann equation, which gives the time-dependent Hubble parameter $H(t)$ as a function of the total energy density $\epsilon(t)$ and the scale factor $a(t)$ of the universe:

$$H^2 = \frac{8\pi G}{3c^2}\epsilon - \left(\frac{c}{R_\circ\, a}\right)^2 \kappa , \tag{3.3}$$

where κ is the constant curvature of space and $R_\circ$ is the constant radius of curvature. The Friedmann equation is one of the most important equations in cosmology. The dimensionless scale factor a is normalized by its value today ($t = t_\circ$):

$$a(t) = \frac{r_{12}(t)}{r_{12}(t_\circ)} , \tag{3.4}$$

so that $a_{\text{now}} = a(t_\circ) = 1$. The Hubble parameter is

$$H = \frac{\dot{a}}{a} , \tag{3.5}$$

and the Hubble constant now is that of Eq. (2.1):

$$H_{\text{now}} \equiv H_\circ = \frac{da}{dt} . \tag{3.6}$$

3.3 The ΛCDM model

The "Lambda cold-dark-matter" (ΛCDM) model is the current standard cosmological model developed to correlate the experimental data after inflation. The model covers time from 1 second after the big bang through today and into the future. The model assumes the entire universe (not all of which is observable) is homogenous and isotropic so that there is no special place and no preferred direction, at least on the scale of galactic clusters. The ΛCDM model is based on the Einstein field equations and the Friedmann equation, assuming in addition that space is Euclidean or flat, not superspherical or hyperbolic. It is assumed that space is filled with energy in the form of radiation, cold dark matter, regular matter, and dark energy, with amounts that vary with time.

For the ΛCDM model, space is assumed to be flat, so $\kappa = 0$ and (3.3) reduces to:

$$H^2(t) = \frac{8\pi G}{3c^2}\epsilon(t)\,. \tag{3.7}$$

Defining a critical energy density ϵ_c,

$$\epsilon_c(t) = \frac{3c^2}{8\pi G}H^2(t)\,, \tag{3.8}$$

allows determination of dimensionless energy density parameters for the components of matter (Ω_m), radiation (Ω_r), and dark energy (Ω_Λ), each of which equals the fraction of that energy in the universe:

$$\Omega_m(t) = \frac{\epsilon_m(t)}{\epsilon_c(t)}\,, \qquad \Omega_r(t) = \frac{\epsilon_r(t)}{\epsilon_c(t)}\,, \qquad \Omega_\Lambda(t) = \frac{\epsilon_\Lambda(t)}{\epsilon_c(t)}\,. \tag{3.9}$$

We know the present value of the parameters from the Planck results[2]: $\Omega_{m0} = 0.315$, $\Omega_{r0} < 0.001$, and $\Omega_{\Lambda 0} = 0.685$. Combining Eqs. (3.2) and (3.8),

$$\Lambda = \frac{3H_\circ^2\Omega_{\Lambda 0}}{c^2} = 1.09 \times 10^{-52}\ \mathrm{m}^{-2}\,. \tag{3.10}$$

The size may be judged by comparing this value to the approximate value expected by the Planck length: $\Lambda \approx (L_p^{-2}) = (10^{-35})^{-2} = 10^{70}\ \mathrm{m}^{-2}$

$$\frac{\Lambda_{\text{actual}}}{\Lambda_{\text{theory}}} = \frac{10^{-52}}{10^{70}} \approx 10^{-120}\,. \tag{3.11}$$

This comparison of the observed small value of the vacuum energy density and theoretical large value of zero-point energy suggested by quantum field theory is called the vacuum catastrophe. Considering that the discrepancy is as high as 120 orders of magnitude, physicists have described the state of affairs as the largest discrepancy between theory and experiment in all of science. This is one of the major unsolved problems in physics. Some physical mechanism must exist that makes the cosmological constant very small[7].

It is easier to judge the size of the cosmological constant written in units of mass density ρ_Λ using (3.2):

$$\rho_\Lambda = \frac{\epsilon_\Lambda}{c^2} = \frac{\Lambda c^2}{8\pi G} = 5.84 \times 10^{-27}\ \mathrm{kg\ m}^{-3}\,, \tag{3.12}$$

which is the mass equivalent to about 3.5 protons per cubic meter for the cosmological constant.

For a flat universe ($\kappa = 0$), the total energy parameter is the sum of the individual energy parameters:

$$\Omega(t) = \Omega_m(t) + \Omega_r(t) + \Omega_\Lambda(t) = 1\,. \tag{3.13}$$

It can be shown[8] that Eq. (3.7) can be written in terms of the Hubble and energy density parameters for now:

$$\frac{da}{dt} = H_0 \left[\frac{\Omega_{r0}}{a^2} + \frac{\Omega_{m0}}{a} + \Omega_{\Lambda 0}\, a^2 + (1 - \Omega_0) \right]^{1/2} . \tag{3.14}$$

In terms of parameters for the ΛCDM model we have $\Omega_{m0} + \Omega_{\Lambda 0} = \Omega_0 = 1$ for a flat universe. We ignore radiation energy density because the radiation-dominated era existed for the relatively short time of 50,000 years after the Big Bang (compared to the 13,800,000,000-year age of the universe). Eq. (3.14) reduces to

$$\frac{da}{dt} = H_0 \left[\frac{(1 - \Omega_{\Lambda 0)}}{a} + \Omega_{\Lambda 0}\, a^2 \right]^{1/2} . \tag{3.15}$$

Rearranging and integrating with respect to a:

$$H_0\, t = \int_{a=1}^{a} \frac{a^{1/2} da}{\sqrt{(1 - \Omega_{\Lambda 0}) + \Omega_{\Lambda 0}\, a^3}} \,. \tag{3.16}$$

This equation can be integrated analytically[8] for $t(a)$:

$$t = \frac{2}{3H_0 \sqrt{\Omega_{\Lambda 0}}} \Bigg(\ln \left[2 \left(\Omega_{\Lambda 0} \sqrt{a^3} + \sqrt{\Omega_{\Lambda 0}} \sqrt{\Omega_{\Lambda 0}(a^3 - 1) + 1} \right) \right] - \ln \left[2 \left(\Omega_{\Lambda 0} + \sqrt{\Omega_{\Lambda 0}} \right) \right] \Bigg) .$$

and its graph is shown on Figure 1 based on

$$H_0 = 67.4 \text{ km s}^{-1} \text{ Mpc}^{-1} , \qquad 1/H_0 = 1.451 \times 10^{10} \text{ yr} , \qquad \Omega_{\Lambda 0} = 0.68$$

The ΛCDM model predicts that the big bang occurred 13.8 billion years ago when the scale factor $a = 0$. The graph demonstrates that we live at a time when the growth rate of the universe da/dt has been nearly linear for several billion years and is now entering a period of accelerated growth in which $d^2a/dt^2 > 0$.

The model also shows the relation between between time and the redshift of observed radiation. The cosmological redshift z between two events is defined by the fractional change in wavelength:

$$z = \frac{\lambda_{\text{obs}} - \lambda_{\text{em}}}{\lambda_{\text{em}}} , \tag{3.17}$$

where λ_{em} is the wavelength of the photon when it was emitted and λ_{obs} is the observed wavelength. The ratio of wavelengths is equal to the ratio of scale factors:

$$\frac{\lambda_{\text{obs}}}{\lambda_{\text{em}}} = \frac{a_{\text{obs}}}{a_{\text{em}}} . \tag{3.18}$$

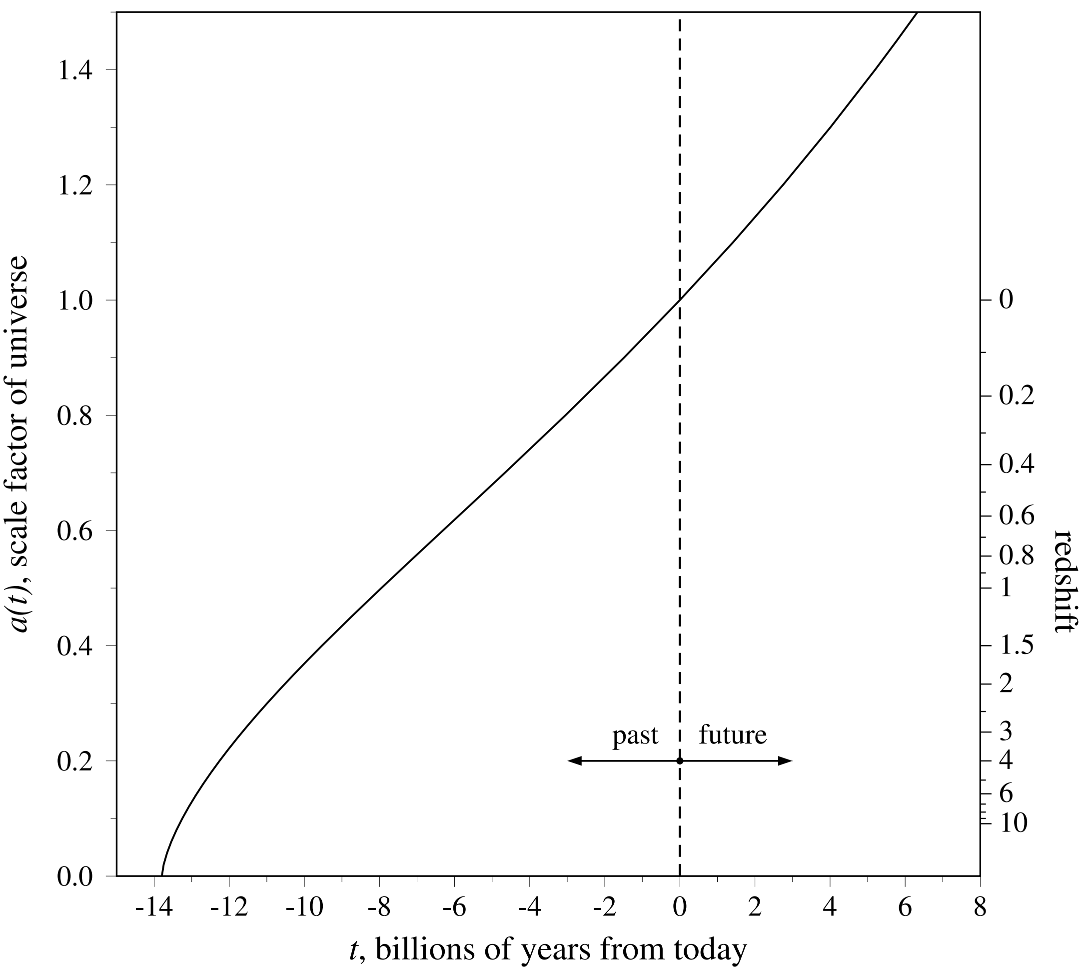

Figure 1: Normalized scale factor a as a function of time from now.

If the observation takes place today ($a_{\text{obs}} = 1$), it follows that the redskip of an object tells the scale factor a_{em} when the photon was emitted:

$$z = \frac{1}{a_{\text{em}}} - 1\,. \tag{3.19}$$

Values for z corresponding to $a(t)$ are shown on the right-hand axis of Figure 1. For example, a photon observed with a redshift of unity was emitted when the universe was one-half its size today or 7.95 billion years ago. Redshifting is also evidence that photons lose energy as they traverse the expanding universe.

We consider next the application of thermodynamics to black holes.

4 Thermodynamics of black holes

Black holes are extremely dense celestial objects that warp spacetime so strongly that no light or matter, once trapped inside its radius or *event horizon*, can escape its gravitational force. The popular image of a black hole is of some huge celestial vacuum cleaner sucking up everything in the universe. This is incorrect; black holes depend entirely on the force of gravity to attract other objects. If our sun were somehow instantaneously collapsed into a black hole (about 6 km in diameter), our solar system would be in darkness but the planets would continue their orbits unchanged. The book by Thorne[9] is a history written by a participant in the quest for understanding black holes, much in the spirit of *The Double Helix* by James Watson describing research into the structure of DNA.

4.1 Entropy, temperature, and lifetimes of black holes

A full theory of black holes will have to await a theory of quantum gravity, but a great deal is already known. A black hole[10] is a gravitationally collapsed star (or combination of millions of stars near the center of a galaxy). If our Sun were collapsed into a black hole, its radius would be only 3 kilometers. The radius of the so-called *event horizon* or point of no return is the Schwarzschild radius R_s:

$$R_s = \frac{2GM}{c^2}\,, \tag{4.1}$$

where M is the total mass of the black hole. The horizon area A of a black hole, where objects are captured, is

$$A = 4\pi R_s^2\,. \tag{4.2}$$

Unlike terrestrial entropy, which is directly proportional to volume, the dimensionless entropy of a black hole is directly proportional to the dimensionless area A of its event horizon[11]:

$$\frac{S}{k_B} = \frac{A}{4L_p^2}\,, \tag{4.3}$$

where L_p = Planck length

$$L_p = \sqrt{\frac{\hbar G}{c^3}} = 1.616 \times 10^{-35}\ \text{m}. \tag{4.4}$$

(4.3) gives the so-called Bekenstein-Hawking entropy, which “ties together notions from gravitation, thermodynamics and quantum theory, and is thus regarded as a window into the as yet mostly hidden world of quantum gravity"[12]. Using (4.4), we can write for (4.3)

$$S = \left[\frac{\pi k c^3}{2hG}\right] A\,. \tag{4.5}$$

Note that just for this equation k stands for the Boltzmann constant (instead of k_B) and the Planck constant h is used (instead of $\hbar$). The reason is that Cambridge professor Stephen Hawking made clear[13] that he would like what is known as Hawking's equation (4.5) carved onto his grave, just as $S = k \log W$ was carved on the tombstone of Boltzmann in Vienna.

Substitution of (4.1), (4.2) and (4.4) into (4.3) gives

$$\frac{S}{k_B} = \frac{4\pi G M^2}{\hbar c}\,. \tag{4.6}$$

The final state of a black hole is determined uniquely by its mass, charge, and angular momentum, i.e., a black hole has no "hair"[14]. The energy of a black hole with no charge or spin is given by Eq. (2.2):

$$E = Mc^2\,. \tag{4.7}$$

With explicit expressions for E and S we can calculate the temperature of the black hole. (Note that $E \neq TS$ because the entropy function is not a homogeneous first-order function of the mass, meaning Euler's theorem on homogeneous functions does not apply.) Since $dE = TdS$,

$$T = \frac{dE}{dS} = \frac{\frac{dE}{dM}}{\frac{dS}{dM}} = \frac{\hbar c^3}{8\pi k_B G M}\,. \tag{4.8}$$

The heat capacity of a black hole can be obtained from the relation

$$C = T\frac{dS}{dT} = T\frac{\frac{dS}{dM}}{\frac{dT}{dM}} = -\frac{8\pi k_B G M^2}{\hbar c} = -2S\,. \tag{4.9}$$

Separation of variables S and T followed by integration gives

$$ST^2 = \frac{\hbar c^5}{16\pi k_B G} = 5.514 \times 10^{39}\ \mathrm{J\ K}\,, \tag{4.10}$$

which applies to any black hole without charge or spin, regardless of its mass. Eq. (4.10) can be verified directly from Eqs. (4.6) and (4.8). This equation derived from the heat capacity of a black hole implies that its entropy is directly proportional to the inverse of its temperature squared.

The Helmholtz free energy of a black hole, F, is

$$F = E - TS = \tfrac{1}{2}Mc^2 = \tfrac{1}{2}E\,, \tag{4.11}$$

so a black hole possesses an enormous capacity to do useful work.

The rate at which an isolated black hole's energy E is radiated away (at absolute vacuum and 0 K) can be found from the Stefan-Boltzmann blackbody radiation formula

$$-\frac{dE}{dt} = -c^2\frac{dM}{dt} = \sigma A T^4\,, \tag{4.12}$$

where the Stefan-Boltzmann constant is

$$\sigma = \frac{\pi^2 k_B^4}{60\hbar^3 c^2} \,. \tag{4.13}$$

Substituting σ, A, and T into (4.12):

$$-\frac{dM}{dt} = \frac{\hbar c^4}{15360\,\pi G^2 M^2} \,. \tag{4.14}$$

Separating variables and integrating from time $0 \to t_{\text{evap}}$ with respect to mass from $M_\circ \to 0$ we obtain the result for the lifetime of a black hole of mass $M_\circ$:

$$t_{\text{evap}} = \frac{5120\,\pi G^2 M_\circ^3}{\hbar c^4} \,. \tag{4.15}$$

These equations are used to calculate properties in Table 1 below.

4.2 Evidence for black holes

Figure 2: Picture of M87* black hole shows shadow of black hole in center surrounded by its yellow and red accretion disk of hot gases.

Fig. 2 is the first picture of a black hole[16], the supermassive M87*, which lies at the center of the elliptical Messier 87 galaxy in the Virgo constellation at a distance of 53 million light years. The event horizon of the black hole is 2.5 times smaller than the shadow it casts in the picture. The accretion disk is asymmetric, indicating an angle of 17° between the light of sight and the spin axis. The resolution needed to take this picture was 20 micro-arc-seconds, which is equivalent to viewing an orange lying on the surface of the moon. The technique used by the Event Horizon Telescope was Very Long Baseline Interferometry (VLBI) at a radio wavelength of 1.3 millimeters using 8 radio telescopes at 6 locations (Antartica, Chile, Mexico, Hawaii, USA, and Spain). The picture was taken in 2017 but two years were required to assemble petabytes of fourier components of radio frequencies synchronized by GPS. The yellow and red colors are used to show the intensity of the 1.3 millimeter radiation, which is clearly greater on the southern part of the accretion ring due to relativistic beaming caused by the bottom part moving toward the observer. The image is consistent with expectations for the shadow of a rotating Kerr black hole as predicted by general relativity.

Table 1 is a list of Schwarzschild properties of black holes. Known black holes are M87*, located in galaxy M87, and Sagittarius A* (abbreviated Sgr A*), located at the center of the

Milky Way at a distance of 25,000 light years. Sgr A* is about 3 orders of magnitude smaller than M87*, but Sgr A* is about 3 orders of magnitude closer, so resolution requirements are about the same for M87* and Sgr A*. Viewing of Sgr A* through the Milky Way is hindered by dust. Properties of a hypothetical small and an intermediate size black hole are listed in Table 1 for comparison.

Table 1: Schwarzschild properties of black holes.

black hole	solar masses	radius, km	earth gravities	entropy, J K^{-1}	T, K	lifetime, years
M87*	6.5×10^9	1.92×10^{10}	2.4×10^2	6.1×10^{73}	9.5×10^{-18}	5.8×10^{96}
Sagittarius A*	4.0×10^6	1.18×10^7	3.8×10^6	2.32×10^{67}	1.54×10^{-14}	1.34×10^{87}
Intermediate	1.0×10^3	2.95×10^3	1.55×10^9	1.45×10^{60}	6.2×10^{-11}	2.10×10^{76}
Small	2.0×10	5.9×10	7.7×10^{10}	5.8×10^{56}	3.1×10^{-9}	1.68×10^{71}

These properties are for Schwarzschild black holes with zero angular momentum. Actual black holes are Kerr black holes which rotate about a central axis and possess rotational energy. An online calculator of these Schwarzschild properties in terms of the mass of the black hole is available[15].

The most notable property of black holes is their enormous value of entropy. We can think of black holes as super-compact trash removal bins that contain most of the entropy of the universe.

The temperature of these black holes is so low at present that they continue to absorb energy from the universe, which is at 2.725 K according to measurements made of the cosmic microwave background (CMB). Only when the temperature of the CMB falls below the temperature of the black hole will it begin to emit radiation to the universe.

5 Conclusions

Experimental behavior for extraterrestrial thermodynamics is different from that usually encountered by chemical engineers. We mention two topics: (1) the ΛCDM model of the universe, for which energy is not conserved, and (2) black holes, for which energy is conserved but the equations differ from classical thermodynamics because of the singularity.

6 Nomenclature

A	horizon area of black hole	m^2
a	normalized scale parameter	dimensionless
$a_\circ$	scale parameter now	$= 1$
c	speed of light	2.9979×10^8 m s^{-1}
d	distance	m
E	energy	J
G	Newtonian gravitational constant	6.6741×10^{-11} m^3 kg^{-1}s^{-2}
$g_{\mu\nu}$	metric tensor	dimensionless
H	Hubble parameter	km s^{-1} Mpc^{-1}
$H_\circ$	Hubble constant	67.4 km s^{-1} Mpc^{-1}
$1/H_\circ$	Hubble time	14.51 Gyr
h	Planck constant	6.6261×10^{-34} J s
$\hbar$	reduced Planck constant	$h/2\pi = 1.0546 \times 10^{-34}$ J s
k	Boltzmann constant, Eq. (4.5)	1.3806×10^{-23} J K^{-1}
k_B	Boltzmann constant	1.3806×10^{-23} J K^{-1}
L_p	Planck length	1.616×10^{-35} m
M	mass	kg
P	pressure	N m^{-2}
$R_{\mu\nu}$	Ricci tensor	m^{-2}
R	curvature scalar	m^{-2}
R_S	Schwarzschild radius	m
S	entropy	J K^{-1}
T	temperature	K
$T_{\mu\nu}$	energy-momentum tensor	m^{-2}
t_{evap}	time for black hole evaporation	s
t	time	s
V	volume	m^3
v	speed	m s^{-1}
z	cosmological redshift of photon	dimensionless
<u>Greek Symbols</u>		
ϵ	energy density	J m^3
κ	spatial curvature	dimensionless
Λ	cosmological constant	1.09×10^{-52} m^{-2}
λ_{em}	wavelength of photon at emission	m
λ_{obs}	wavelength of photon at detection	m
Ω	total density parameter	dimensionless
ρ	mass density	kg m^{-3}
σ	Stefan-Boltzmann constant	5.6704×10^{-8} W m^{-2} K^{-4}
<u>Subscripts</u>		
m	matter (baryonic and dark)	
r	radiation	
Λ	dark energy	
$\circ$	refers to t = now	

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