SPECIAL RELATIVITY

ALAN L. MYERS

Abstract. The theory of special relativity assumes that the velocity of light is a universal speed limit. There is no universal clock in spacetime. The laws of physics such as $F = ma$ are independent of the relative motion of an inertial frame of reference. Time is a fourth dimension in addition to the three spatial dimensions that we move around in. The implications of special relativity are that moving clocks slow down and that moving objects shrink in the direction of motion. Two events that occur at different times for one person can be simultaneous for a second person in motion relative to the first. Making sense of special relativity requires an understanding of Lorentz transformations, time dilation, and Fitzgerald-Lorentz contraction. The Minkowski diagram provides a geometric interpretation of events in spacetime. A sample diagram shows how two inertial frames in relative motion exhibit time dilations and contractions in *both* directions. Several other examples are provided. This short development shows how so-called paradoxes vanish when it is understood that the version of events in each inertial frame is self-consistent with experiment. Different frames of reference are in complete agreement about relative velocity, events in spacetime, and the laws of physics.

1. Galilean transformation of coordinates

Consider two coordinate systems S and S' moving relative to each other in the x-direction. Specifically, coordinate system S' is moving to the right in the direction of the positive x-axis at constant velocity $v_{S'/S}$ relative to S. Alternatively, coordinate system S is moving to the left at constant velocity $v_{S/S'}$ relative to S' so that $v_{S/S'} = -v_{S'/S}$. Relative motion occurs only along the x-axis; y and z coordinates coincide in S and S'. Initial conditions at $t = 0$ are $x = x' = 0$.

The Galilean coordinate transformations are:

$$
\begin{aligned}
x' &= x - vt \\
x &= x' + vt \\
y &= y' \\
z &= z' \\
t &= t'
\end{aligned}
$$

Velocities are additive:

$$
\begin{aligned}
u'_x &= u_x - v \\
u_x &= u'_x + v
\end{aligned}
$$

and acceleration is an invariant

$$a'_x = a_x$$

Notice that $v_{S'/S}$ is calculated by setting $x' = 0$ (the location of the origin of S'):

$$v_{S'/S} = \frac{x}{t} = v \qquad \text{(for } x' = 0\text{)}$$

2. Derivation of Lorentz transformations

The key assumption is that the velocity of light (c) is the same in all coordinate systems. Coordinates (t and x) are required to describe an event in spacetime. Each inertial system has a different but self-consistent version of length and time measurements. Certain invariants are independent of coordinates: these include the velocity of light, the spacetime interval $(-t^2 + x^2)$, the relative velocity of pairs of observers, and the location of events in four-dimensional spacetime.

The most general form which retains linearity for transformation of coordinates is:

$$x' = ax - bt \tag{1}$$

$$x = ax' + bt' \tag{2}$$

where a and b are functions of v. Time is no longer absolute and in general $t \neq t'$. As before, the relative velocity $v_{S'/S}$ is calculated by setting $x' = 0$:

$$v_{S'/S} = \frac{x}{t} = \frac{b}{a} \qquad \text{(for } x' = 0\text{)}$$

Similarly

$$v_{S/S'} = \frac{x'}{t'} = -\frac{b}{a} \qquad \text{(for } x = 0\text{)}$$

For $v_{S'/S} \equiv v$

$$b = a\,v \tag{3}$$

Consider next a light signal emitted at $t = t' = 0$ when the origins of the coordinate systems coincide ($x = x' = 0$). The propagation of the signal in the S and S' coordinate systems is

$$x = ct \tag{4}$$

$$x' = ct' \tag{5}$$

because the basic assumption of the theory of special relativity is that the velocity of light (c) is independent of the relative motion of the system of coordinates in which it is measured.

The rest is algebra. Substituting Eqs. (4) and (5) into Eqs. (1) and (2) gives:

$$ct' = (ac - b)t \tag{6}$$

$$ct = (ac + b)t' \tag{7}$$

Substitution of Eq. (7) into Eq. (6) yields:

$$c\,t' = (ac - b)\left[\frac{ac + b}{c}\right]t'$$

$$c^2 = a^2c^2 - b^2$$

Using Eq. (3)

$$c^2 = a^2(c^2 - v^2)$$

$$a = \frac{1}{\sqrt{1 - v^2/c^2}} \equiv \gamma \tag{8}$$

$$b = av = \gamma v \tag{9}$$

Substitution of Eqs. (8) and (9) into Eqs. (1) and (2) gives the Lorentz transforms for x:

$$\boxed{x' = \gamma(x - vt)} \tag{10}$$

$$\boxed{x = \gamma(x' + vt')} \tag{11}$$

The transformation for time is obtained by substituting x' from Eq. (10) into (11):

$$x = \gamma[\gamma(x - vt) + vt']$$

$$\frac{x}{\gamma} = \gamma x - \gamma vt + vt' \implies vt' = \gamma vt + \left(\frac{1}{\gamma} - \gamma\right)x \implies t' = \gamma\left[t + \left(\frac{1}{\gamma^2} - 1\right)\frac{x}{v}\right]$$

From Eq. (8):

$$\left(\frac{1}{\gamma^2} - 1\right) = 1 - \beta^2 - 1 = -\beta^2 = -\frac{v^2}{c^2}$$

so

$$\boxed{t' = \gamma\left(t - \frac{vx}{c^2}\right)} \tag{12}$$

Similarly:

$$\boxed{t = \gamma\left(t' + \frac{vx'}{c^2}\right)} \tag{13}$$

Reduction of these to the Galilean $t = t'$ requires that $x << ct$ as well as $v << c$.

3. Relativistic addition of velocities

Start with the basic equations for transformation of coordinates:

$$\begin{aligned} x &= \gamma(x' + vt') \\ y &= y' \\ t &= \gamma\left(t' + \frac{vx'}{c^2}\right) \end{aligned}$$

Suppose that an object has velocity component u'_x, u'_y as measured in S'. From the definition of velocity:

$$u'_x = \frac{dx'}{dt'} \qquad u'_y = \frac{dy'}{dt'}$$

Assuming that S' has the velocity v relative to S, what are the components of the velocity in the S frame? By differentiation:

$$\begin{aligned} dx &= \gamma(u'_x + v)\, dt' \\ dy &= u'_y\, dt' \\ dt &= \gamma\left(1 + \frac{vu'_x}{c^2}\right) dt' \end{aligned}$$

Therefore:

$$\boxed{u_x = \frac{dx}{dt} = \frac{u'_x + v}{1 + vu'_x/c^2}} \tag{14}$$

$$\boxed{u_y = \frac{dy}{dt} = \frac{u'_y/\gamma}{1 + vu'_x/c^2}} \tag{15}$$

Eq. (14) is the relativistic law for addition of velocities. If both u'_x and v are small compared to c, the straightforward addition of velocities for the kinematics of everyday life works. Note that $u_x \leq c$. Also $u_x = c$ when $u'_x = c$, as required if the speed of light is the same in all inertial frames.

Example: For $u'_x = v = 0.5$, $u_x = 0.8$ (dimensionless velocities).

4. Invariance of spacetime interval

Set $c = 1$ in the Lorentz transformations

$$x' = \gamma(x - vt)$$
$$t' = \gamma(t - vx)$$

where v is dimensionless and t has units of length.

$$(x')^2 = \gamma^2(x^2 - 2xtv + t^2v^2)$$
$$(t')^2 = \gamma^2(t^2 - 2xtv + x^2v^2)$$

so that

$$(t')^2 - (x')^2 = \frac{t^2 - x^2 + v^2(x^2 - t^2)]}{1 - v^2} = t^2 - x^2.$$

The spacetime interval between two events is independent of the coordinate system.

5. Covariance of a physics equation

Newton's second law is covariant with respect to rotation, which means that the law has rotational symmetry. Consider the two sets of axes xyz and XYZ in Figure 1. The z and Z axes coincide at the origin and are perpendicular to the page.

Transform the xyz set of coordinates to the XYZ set by CCW rotation about the z axis through angle θ.

Cartesian and polar coordinates of $\mathbf{R}$ are related by:

$$x = R\cos(\theta + \phi) \qquad y = R\sin(\theta + \phi)$$
$$X = R\cos\phi \qquad Y = R\sin\phi$$

From trigonometry:

$$\cos(\theta + \phi) = \cos\theta\cos\phi - \sin\theta\sin\phi$$
$$\sin(\theta + \phi) = \sin\theta\cos\phi + \cos\theta\sin\phi$$

Substituting:

$$x = R\cos\theta\cos\phi - R\sin\theta\sin\phi$$
$$y = R\sin\theta\cos\phi + R\cos\theta\sin\phi$$

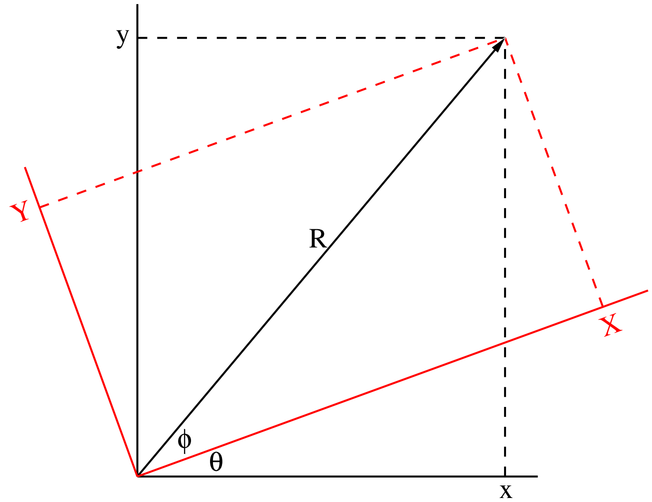

FIGURE 1. Rotation of axes for vector **R**.

$$x = X\cos\theta - Y\sin\theta$$
$$y = X\sin\theta + Y\cos\theta$$

Invert with Cramer's rule. The value of the determinate of the denominator is $\sin^2\theta + \cos^2\theta = 1$ so:

$$X = \begin{vmatrix} x & -\sin\theta \\ y & \cos\theta \end{vmatrix} = \quad x\cos\theta + y\sin\theta$$
$$Y = \begin{vmatrix} \cos\theta & x \\ \sin\theta & y \end{vmatrix} = -x\sin\theta + y\cos\theta$$

This is written as a rotation transformation matrix:

$$\begin{pmatrix} X \\ Y \\ Z \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} \tag{16}$$

In component notation, the rotational transformation from $\mathbf{v}$ to $\mathbf{v}'$ using rotation matrix $[R]$ is:

$$v'_i = \sum_{j=1}^{3} [R]_{ij} v_j = [R]_{i1} v_1 + [R]_{i2} v_2 + [R]_{i3} v_3$$

Newton's second law is $\mathbf{F} = m\mathbf{a}$ or $F_i = ma_i$ in component form.

$$F'_i - ma'_i = \sum_j [R]_{ij} F_j - m \sum_j [R]_{ij} a_j = \sum_j [R]_{ij}(F_j - ma_j) = 0$$

Thus the validity of Newton's second law in a system S implies its validity in any other system S' related to S by a rotation. We see that if a physics equation can be written as a tensor equation, it is covariant (its form is unchanged) and automatically respects rotational symmetry. Covariance of a physics equation means that the laws of physics are the same in different reference frames.

6. Lorentz transformation from rotation of 4D spacetime coordinates

Consider the equations for transforming coordinates from S to S' in 2D. In this section, time has units of length and $c = 1$. From Eq. (16):

$$\begin{aligned} y' &= -x\sin\theta + y\cos\theta \\ x' &= \quad x\cos\theta + y\sin\theta \end{aligned}$$

Let the y axis be the it axis to form a complex plane:

$$\begin{aligned} it' &= -x\sin\theta + it\cos\theta \\ x' &= \quad x\cos\theta + it\sin\theta \end{aligned}$$

Using $\cos\theta = \cosh i\theta$ and $\sin\theta = -i\sinh i\theta$ and defining $\psi \equiv i\theta$:

$$t' = x\sinh\psi + t\cosh\psi \tag{17}$$

$$x' = x\cosh\psi + t\sinh\psi \tag{18}$$

Along the t' axis at $x' = 0$ we have:

$$\frac{x}{t} = -\frac{\sinh\psi}{\cosh\psi} = -\tanh\psi = v$$

Using this equation and the two identities:

$$\cosh^2\psi - \sinh^2\psi = \cosh\psi\sqrt{1 - \frac{\sinh^2\psi}{\cosh^2\psi}} = 1$$

we find

$$\cosh\psi = \frac{1}{\sqrt{1-v^2}} = \gamma\;; \qquad \sinh\psi = -\gamma v$$

From Eqs. (17) and (18), the 2D portion of the 4D coordinate transformation is:

$$\begin{pmatrix} t' \\ x' \end{pmatrix} = \gamma \begin{pmatrix} 1 & -v \\ -v & 1 \end{pmatrix} \begin{pmatrix} t \\ x \end{pmatrix} \tag{19}$$

This is the matrix form of the Lorentz transform, Eqs. (10) and (12). Considering the time-axis to be imaginary, it has been shown that its rotation by angle ψ is equivalent to a Lorentz transformation of coordinates.

This derivation is remarkable but in general it is not useful to consider the time axis to be imaginary. In relativity, the time axis is distinguished from the $\{x, y, z\}$ axes only by its signature is 4-D spacetime. The hyperbolic functions also arise naturally in accelerated motion.

7. Galilean symmetry

A Galilean transformation goes from one inertial frame S with coordinates x_i to another inertial frame S' with coordinates x_i'. Interest is focused on inertial frames with the same orientation $[R]_{ij} = \delta_{ij}$. If the relative velocity of the two frames is $\mathbf{v}$:

$$x' = x - vt \qquad y' = y \qquad z' = z \qquad t' = t \tag{20}$$

This kind of transformation is called a boost. It is understood that S and S' coincide at $t = 0$. Differentiating Eq. (20) with respect to t:

$$u' = u - v \tag{21}$$

$$a' = a \tag{22}$$

A physics equation such as Newton's law of gravitation does not change under a Galilean transformation:

$$\mathbf{g} = \frac{GM\hat{\mathbf{r}}}{r^2}$$

because the vector $\mathbf{r}$ and the acceleration $\mathbf{g}$ are unchanged by the transformation.

8. Electrodynamics and Lorentz symmetry

Maxwell's equations are not covariant under the Galilean transformation. The propagation speed of electromagnetic waves is a constant:

$$c = \sqrt{\frac{1}{\mu_\circ \epsilon_\circ}}$$

Since $\epsilon_\circ$ and $\mu_\circ$ are constants in Coulomb's and Ampere's law, c must be the same in all reference frames. This constancy of the speed of light violates the Galilean velocity addition rule. Maxwell's equations are covariant under the Lorentz boost transformation:

$$x' = \gamma(x - vt), \qquad y' = y, \qquad z' = z, \qquad t' = \gamma(t - \frac{v}{c^2}x). \tag{23}$$

where

$$\gamma = \frac{1}{\sqrt{1-\beta^2}}, \qquad \beta = \frac{v}{c}.$$

This transformation of the S system into the S' system can be inverted by noting that while the S' system is moving at v in the S coordinates, the S system is moving at $-v$ in the S' coordinates. All we need to do is interchange (x, t) with (x', t') and v with $-v$:

$$x = \gamma(x' + vt'), \qquad t = \gamma(t' + \frac{v}{c^2}x') \tag{24}$$

An electric charge at rest gives rise to an electric field, but no magnetic field. The same situation when viewed by a moving observer is seen as a charge in motion, which produces electric and magnetic fields. Different inertial observers will find different electric and magnetic fields. Under the Lorentz transformation, not only space and time coordinates will change, but also electromagnetic fields and source charge and currents will change. However, Maxwell's equations are covariant under the Lorentz transformation.

9. The basic postulates of special relativity

Principle of relativity. Physics laws have the same form in every inertial frame of reference. No physical measurement can reveal the absolute motion of an inertial frame of reference.

Constancy of the speed of light This second postulate is consistent with the first one, as the constancy of the speed of light is a feature of electrodynamics and hence a law of physics.

Einstein pointed out that a new kinematics was required: different time coordinates are required for different coordinate frames when the speed of signal transmission is not infinite. Einstein emphasized that the definition of time was ultimately based on the notion of simultaneity. Since all signals travel at a finite speed, simultaneity becomes a relative concept.

10. Relativity of spatial equilocality

Two events occurring at the same location in an inertial frame S' are termed "equilocal". $\Delta x' = x'_2 - x'_1 = 0$ and $t'_2 - t'_1 \neq 0$. Even under a Galilean transformation $\Delta x = v\Delta t \neq 0$ although $\Delta x' = 0$. Imagine a light bulb on a moving train emitting two flashes of light. On the train, the flashes are equilocal, but to the observer on the train platform the flashes appear to be emitted at different locations.

11. Relativity of simultaneity

Consider the emission of a light pulse from the midpoint of a railcar towards the front and back end of the railcar. Let the length of the railcar be L' in S'. For the observer on the railcar, the light pulse will arrive at the front and back ends of the railcar simultaneously. The emission event has coordinates $x' = 0$, $t' = 0$. The arrival events are $(-L'/2, t'_1)$ and $(L'/2, t'_2)$ at the back and front ends of the railcar, respectively. The arrival events are simultaneous in S' because $t'_1 = t'_2 = L'/2c$.

Using Eq. (13):

$$t_1 = \frac{\gamma L'}{2c}(1 - \frac{v}{c})$$
$$t_2 = \frac{\gamma L'}{2c}(1 + \frac{v}{c})$$
$$\Delta t = t_2 - t_1 = \frac{\gamma L' v}{c^2}$$

This may be written:

$$c\,\Delta t = \gamma \left(\frac{v}{c}\right) L' \tag{25}$$

If two events separated by distance L' are simultaneous in one reference frame, in the other frame one event will lag behind the other event according to Eq. (25). From the stationary frame, the event located in the *rear* along the direction of motion occurs first.

Spatial equilocality and time simultaneity are obviously similar; yet, while relativity of the first is considered to be obvious, the second is counter-intuitive.

12. Synchronizing clocks

For a given inertial frame, synchronize clocks with a master clock at the origin by sending out a signal from the origin at $t = 0$. When the clock receives the signal at a distance r from the origin, it should be set to $t = r/c$. Equivalently, synchronization of any two clocks can be checked by sending out light flashes from these two clocks at a given time. If the two flashes arrive at their midpoint at the same time, the clocks are synchronized.

If the time interval between two flashes of light measured by a clock at one location is the same for another clock at a second location, the two clocks are synchronized.

Or, simply stated but impractical to execute, synchronize clocks A and B with a third clock C carried *very slowly* from clock A to clock B.

13. Time dilation

Dilation of the pupil of an eye means to enlarge the pupil. In mathematical terminology, dilation refers to both enlargement and reduction in size. In relativity, time dilation means that a moving clock slows down. The period Δt between successive clicks increases for a clock moving relative to a reference frame. As the period increases, ticking frequency decreases and the clock "slows down".

A ticking clock marks time intervals or periods $\Delta t'$ in its own rest frame S'. $\Delta t'$ is called "proper time" or "eigenzeit". In the clock's rest frame, its position is unchanged ($\Delta x' = 0$). However, an observer in frame S in motion with respect to S' sees a moving clock and the interval in frame S will be observed as a longer interval according to Eq. (13):

$$\Delta t = \gamma \Delta t'$$

Thus we say that a moving clock (i.e., moving with respect to the S system) runs slow. The time registered by a clock traveling between two events depends upon its path in spacetime.

The slowing down of moving clocks is not some kind of illusion. A moving clock really slows down. This was verified experimentally in 1971 by Hafele and Keating, who flew an atomic clock around the earth on an airplane and compared its number of ticks with a clock on the ground. A clock flying east around the earth moves faster than a clock on the ground (relative to the earth's center) and therefore slows down relative to the grounded clock. The difference in time registered by the moving and ground clocks was of the order of nanoseconds.

14. Fitzgerald-Lorentz contraction

The length L of an object moving in frame S, compared to the length L' of the object measured in its own rest frame S' with a ruler, appears to be shortened. Consider measuring the length of a railcar with a clock fixed on the ground. A ground observer in frame S, watching the train in frame S' move past the clock with speed v, can calculate the car's length by reading off the times when the front and back ends of the railcar pass the clock:

$$L = v(t_2 - t_1) = v\,\Delta t \tag{26}$$

An observer in frame S' on the railcar knows the relative velocity v and can make a similar calculation

$$L' = v(t'_2 - t'_1) = v\,\Delta t' \tag{27}$$

but the observer on S' doesn't know the value of $\Delta t'$ because the clock on the ground is moving relative to the railcar. Since Δt is the eigenzeit or proper time:

$$\Delta t' = \gamma\,\Delta t \tag{28}$$

From Eqs. (26), (27), and (28)

$$\frac{L}{L'} = \frac{\Delta t}{\Delta t'} = \frac{1}{\gamma} \tag{29}$$

so

$$L = \frac{L'}{\gamma} \tag{30}$$

L' is the proper length of the railcar, which can be measured on the railcar with a ruler. L is the contracted length of the moving railcar.

Time dilation and length contraction ensure that the relative velocity v is the same in both reference frames. From Eqs. (26) and (27):

$$\frac{L}{\Delta t} = \frac{L'}{\Delta t'} = v$$

15. Metric of Spacetime

$$ds^2 = -c\,dt^2 + dx^2 + dy^2 + dz^2$$

Compare to the metric of a spherical surface:

$$ds^2 = R^2(d\theta^2 + \sin^2\theta\, d\phi^2)$$

where ϕ is the azimuthal angle (0 to 2π) and θ is the polar angle (0 to π).

16. Principle of Causality

In physics, a cause always precedes its effect. In Newtonian mechanics, absolute time serves to separate the past, present, and future. In special relativity (SR), the order of events varies with the inertial system. The fundamental postulate of SR is that the speed of light is the same in all inertial systems. Objects moving *faster* than the speed of light violate the principle of causality. Consider a spaceship traveling from earth to Proxima Centauri. Its velocity is $v = x/t$, where x is the distance (4.24 light-years) and t is the travel time, all in earth coordinates. The Lorentz transform for the spaceship time coordinate t' is:

$$t' = \gamma\left(t - \frac{v}{c^2}x\right)$$

Substituting $t = x/v$:

$$t' = \frac{\gamma x}{v}\left[1 - \left(\frac{v}{c}\right)^2\right] \tag{31}$$

A value of $t' < 0$ would violate the principle of causality because the spaceship would arrive at Proxima Centauri before it left the earth at $t' = 0$. Eq. (31) sets a speed limit of $v = c$ for non-violation of the principle of causality.

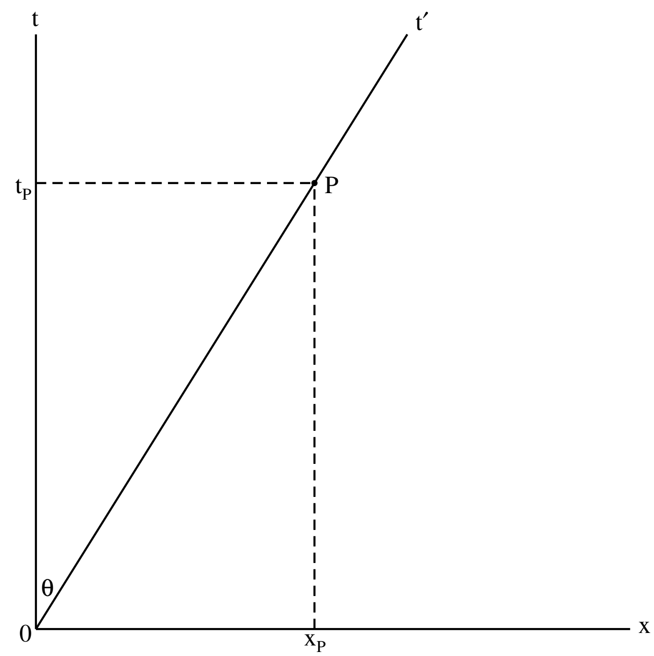

FIGURE 2. World line OP for system S' moving at constant velocity relative to system S.

17. Spacetime (Minkowski) Diagram

Rules of construction:
1. t and x axes scaled in the same units of length: (ct) and x.
2. Angle of t' axis with the vertical and the angle of the x' axis with the horizontal is $\theta = \tan^{-1}(v/c)$.
3. Scale factor for t' and x' axes is $1/\sqrt{\cos(2\theta)}$.

In special relativity, time is measured in units of length by setting the speed of light equal to unity ($c = 1$). Time in MKS units can always be recovered by dividing the time (measured in units of length) by c. Thus a "time" of one light-year is one year in MKS units; a "time" of one meter is equal to $1/c = 3.336 \times 10^{-9}$ s in MKS units. In the following description of Minkowski diagrams, velocity $c = 1$ and velocity v is dimensionless.

The rules for constructing a Minkowski diagram are derived from the axioms of plane geometry. On Fig. 2, the abscissa is distance x and the ordinate is time t (both measured in units of length) for system S. For a system S' moving at velocity v relative to S, oblique coordinates are required for distance x' and time t'. The convention is to use the same origin ($x = x' = 0$) and ($t = t' = 0$) for both systems. The S' system usually is moving to the right relative to S at constant velocity v; in this case the t' axis (at $x' = 0$) has a positive slope with $v = x_P/t_P$. Referring to Fig. 2, $\tan\theta = v$. The t'-axis is called the world line for the moving point located at $x' = 0$.

A point on a Minkowski diagram such as point P on Fig. 2 is called an *event*, located at (x, t) in the S system and at (x', t') in the O' system. We have the t' axis but where is the x' axis? Simultaneous events in S lie on a horizontal line. The next step is to find the slope of lines for simultaneous events in S'. Consider the train OE on Figure 3; at time $t = 0$ the back of the train is located at the origin and the front of the train is located at point E. Assume that train's velocity $v = 0.88$ so that $\theta = \tan^{-1}(0.88) = 30°$. The three parallel world lines of the back, middle, and front of the train are OA, DB, and EC, respectively. Event D is the emission of a flash of light, the trajectory of which in the S system is represented by the dashed lines DA and DC. These lines representing photons traveling at the speed of light make an angle of $\tan^{-1}(1) = 45°$ with the horizontal and vertical axes. Events A (arrival of photons at rear of train) and event C (arrival of photons at front of train) are simultaneous in the S' system because they signal the arrival of a flash of light emitted from the center of the train.

The three world lines are parallel and equidistant, with $AB = BC$. ADC is a right triangle. The converse of Thales' theorem is that the center of a circumcircle of a right triangle lies on its hypotenuse. AC is the diameter of the circumcircle and point B is its center because $AB = BC$. $DB = BC$ because both lines are radii of the circumcircle. DBC is an isosceles triangle with equal base angles (α). Because $\alpha + \theta = 45°$ at point D, the same equality applies at point C and line ABC makes an angle θ with the horizontal. Points A and C are simultaneous events in system S'. It has been proven that two events occurring on a straight line making an angle θ with the horizontal are simultaneous. Since the only criterion for simultaneous events is the angle θ of the line connecting them with the horizontal, it may be concluded that *all* points on the line represent events occurring at the same time in the t' system. Specifically, the x' axis corresponding to events at $t' = 0$ passes through the origin at angle θ with the horizontal.

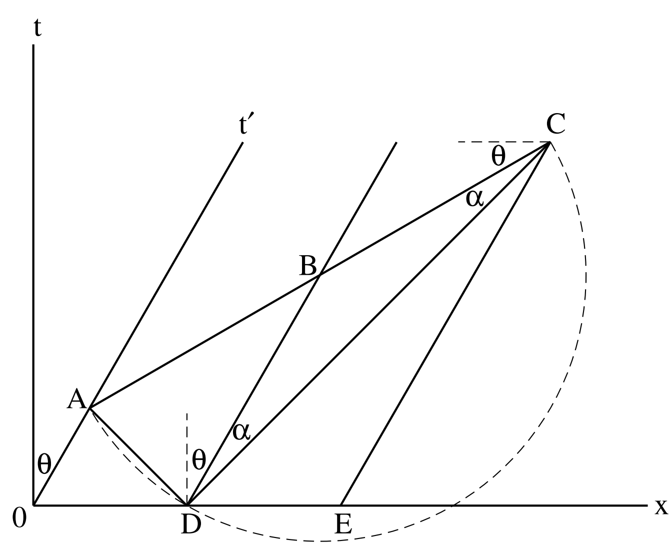

FIGURE 3. Geometric proof for slope of x' axis on Minkowsky diagram.

Given the relative velocity v of the S and S' systems, the t' axis makes an angle θ with the vertical and the x' axis makes the same angle θ with the horizontal, with $\theta = \tan^{-1} v$, as illustrated on Figure 5.

The final step is to determine the scales for the t' and x' axes forming an oblique coordinate system. The scales for the t and x axes in the S system are identical (see Figure 5) so we might expect the same result for the S' system. Figure 4 shows a light ray emitted at the origin and the event of its arrival at point P, with coordinates (x_P, t_P) in the S system and (x'_P, t'_P) in the S' system. The slope of OP is 45° because $\theta = \tan^{-1}(v) = \tan^{-1}(1) = 45°$. Since OA makes an angle θ with the vertical and OB makes an angle θ with the horizontal, it follows that $\alpha = 45° - \theta =$ angle POB = angle POA. Line OP biserts angle AOB so $OAPB$ is a rhombus with four equal sides; specifically, $OA = OB$. The dimensionless speed of the light ray in the S' system is $v = x'_P/t'_P = 1$ so $x'_P = t'_P$. Since $OA = OB$, the scaling factors for the x' and t' axes are identical. Because both pair of axes have the same stretching factor (x'/x and t'/t) it is enough to deduce the factor for just one pair. The scaling factor will be derived for the x' axis.

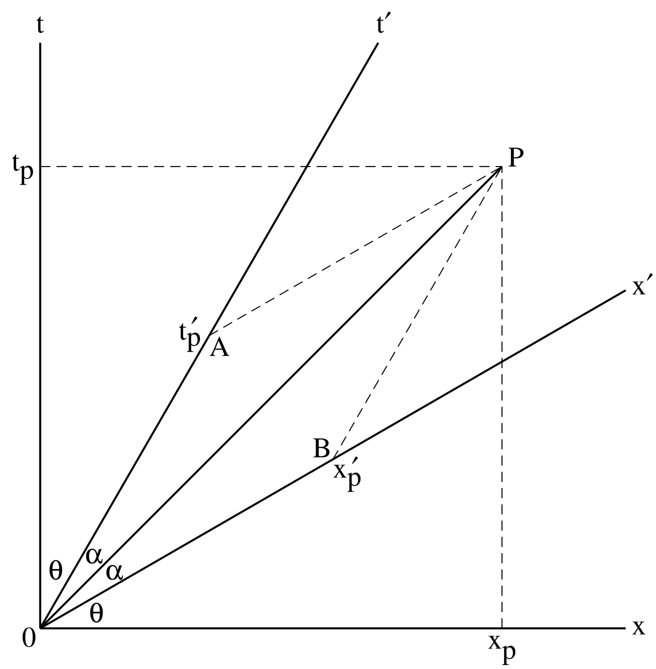

FIGURE 4. Equality of scale factors on t' and x' axes of Minkowski diagram.

On Figure 5 units (0,1,2,3...) are marked on the x and t axes of system S and the objective is to find the scaling factor for units in the x' and t' axes of system S'. The x'-axis makes an angle θ with the horizontal and the t'-axis makes the same angle θ with the vertical. Draw the hyperbola $x^2 - t^2 = 1$ in the O system of coordinates; the intersection of the hyperbola with the x' axis at point P marks a length of 1 unit on the x' axis in the O' system of coordinates.

A trigonometric formula will be needed later:

$$\cos 2\theta = \frac{1 - \tan^2\theta}{1 + \tan^2\theta} \tag{32}$$

This follows from

$$\cos 2\theta = \cos^2\theta - \sin^2\theta = \cos^2\theta(1 - \tan^2\theta)$$

and

$$\sin^2\theta + \cos^2\theta = 1 \qquad \Longrightarrow \qquad \cos^2\theta = \frac{1}{1 + \tan^2\theta}$$

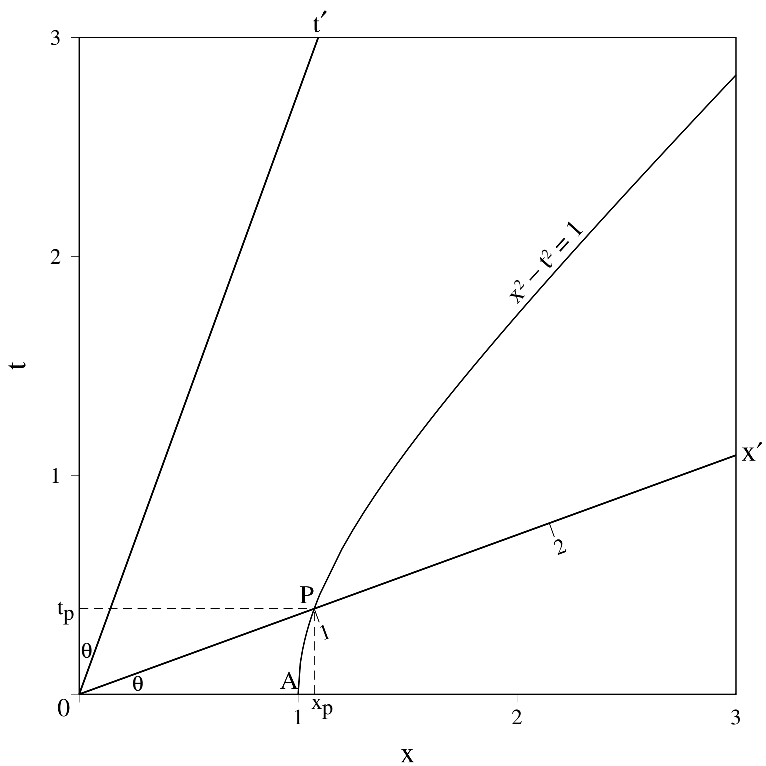

FIGURE 5. Scale factor OP/OA for x' and t' axes of Minkowski diagram.

The Lorentz transformations are:

$$x_P = \gamma(x' + vt')$$
$$t_P = \gamma(t' + vx')$$

Point P is located at $t' = 0$ and $x' = 1$ so

$$x_P = \gamma$$
$$t_P = \gamma v$$

In terms of S coordinates, $OA = 1$ so the scale factor OP/OA using the Pythagorean theorem is

$$\frac{OP}{OA} = \sqrt{x_P^2 + t_P^2} = \sqrt{\gamma^2 + \gamma^2 v^2} = \gamma\sqrt{1+v^2} = \frac{\sqrt{1+v^2}}{\sqrt{1-v^2}}$$

Using $v = \tan\theta$ and Eq. (32)

$$\boxed{\text{scale factor} = \frac{OP}{OA} = \frac{1}{\sqrt{\cos 2\theta}}}$$

18. Numerical examples

Some of the following numerical examples are *gedanken* or thought experiments involving trains moving close to the earth at speeds near the speed of light, while ignoring practical "details" such as the effect of friction. Time is expressed as a length (ct) and velocity is dimensionless (v/c).

18.1. **Relativistic train passing through a tunnel.** The earth observation is a 30 m train passing through a tunnel of the same length at a velocity of 0.8 so that $\gamma = 1/\sqrt{1-0.8^2} = 5/3$. The Minkowski diagram is shown on Figure 6. The blue area is the trajectory or world region of the train, analogous to a world line for a point. The yellow area is the tunnel and the green area is the intersection of the train and the tunnel, which identifies four particular events. Event A is the front of the train emerging from the tunnel exit and event B is the disappearance of the rear of the train into the tunnel entrance. Event C is the front of the train at the tunnel entrance and event D is the rear of the train emerging from the tunnel exit.

Coordinates for the four events (A,B,C,D) are in Table 1. The transformation of coordinates from (t, x) to (t', x') obeys Eqs. (10) and (11). The spacetime interval is $\sqrt{|x^2 - t^2|} = \sqrt{|(x')^2 - (t')^2|}$. Earth coordinates (t, x) are easily read from Figure 6. The train time (t') at any point is found by drawing a line through the point parallel to the x'-axis; the value of t' is read at the intersection of the parallel line with the t'-axis (for $x' = 0$). Lines of constant location x' are drawn for $x' = 0$ and $x' = -50$. The train location (x') at any point is found by drawing a line through the point parallel to the t'-axis; the value of x' is read at the intersection of the line with the x'-axis.

Table 1. Earth and train coordinates for train passing through tunnel at relativistic speed.

event	location	earth		train		interval
		x	t	x'	t'	
C	front of train at tunnel entrance	-30	-37.5	0	-22.5	22.5
A	front of train at tunnel exit	0	0	0	0	0
B	rear of train at tunnel entrance	-30	0	-50	40	30
D	rear of train at tunnel exit	0	37.5	-50	62.5	37.5

The earth version of the encounter is that a train 30 m in length passes through a tunnel of the same length. The train version is that a train 50 m long passes through a tunnel 18 m in length. Thus the train does not fit inside the tunnel and when the

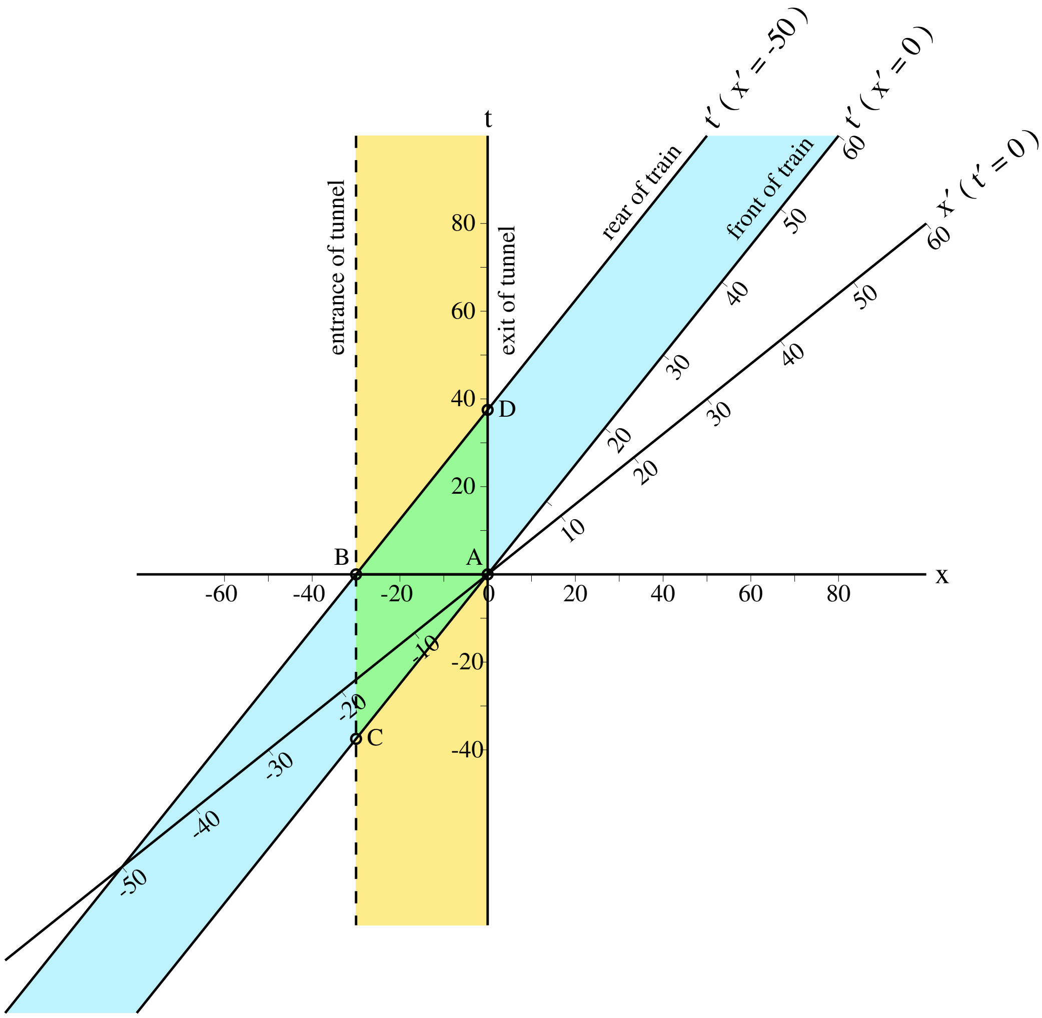

FIGURE 6. Spacetime diagram of train passing through tunnel at $0.8\,c$. Event A is the front of the train located at the tunnel exit and event B is the rear of the train located at the tunnel entrance. The angle between the t and t' axes is $\theta = \tan^{-1} v = \tan^{-1} 0.8 = 38.66°$. The scale factor is $1/\sqrt{\cos(2\theta)} = 2.1344$.

front of the train reaches the tunnel exit, the rear 32 m of the train extends outside the tunnel entrance. It takes $t = d/v = 32/0.8 = 40$ m of additional time for the rear of the train to reach the tunnel entrance, so in the train version, event B occurs 40 m after event A.

The proper length of the tunnel (30 m), measured on the earth with a laser ruler, contracts to $30/\gamma = 18$ m in train coordinates. The proper length of the train (50 m), measured on the train with a laser ruler, contracts to $50/\gamma = 30$ m in earth coordinates.

The proper train-passage time of 37.5 m, measured with a stationary clock located on the earth at the entrance (or exit) of the tunnel, dilates to $37.5\gamma = 62.5$ m in train coordinates. The proper tunnel-transit time of 22.5 m, measured with a stationary clock located on the front (or rear) of the train, dilates to $22.5\gamma = 37.5$ m in earth coordinates.

The relative velocity of the train and earth is given by the length of the tunnel divided by the tunnel-transit time. For the earth, $v = 30/37.5 = 0.8$ and for the train, $v = 18/22.5 = 0.8$. The systems agree upon their relative velocity, as required. The relative velocity is also equal to the length of the train divided by the train-passage time: $50/62.5 = 0.8$.

In the earth version, events A and B are separated by a distance $L = 30$ m and occur simultaneously. According to Eq. (25), in the train version the events A and B are not simultaneous, but are separated by a time interval

$$\Delta t = \gamma v L = \frac{5}{3}(0.8)(30) = 40 \text{ m}$$

For the Minkowski diagram in Figure 6, there are two entirely different versions of events, the earth version and the train version. The earth version sees a train contracted in length from the measurement made on the train with a ruler, and the train version sees a tunnel contracted in length compared to the earth measurement with a ruler. The earth and train coordinate systems cannot even agree on the temporal order of events: the earth clock indicates that events A and B are simultaneous, while the train clock indicates that event B occurs after event A. Nevertheless, the train and earth versions of what happened are self-consistent. The inertial systems agree upon the locations of the events in four-dimensional spacetime. The earth and train inertial systems agree upon their mutual relative velocity ($v = 0.8$). The spacetime intervals of events in Table 1 are invariants.

18.2. **Rocket Speed Calibration.** System S is a straight 4 km track with synchronized clocks at the start and finish lines. The rocket, system S', was timed at 5 km on the 4 km track, a velocity of 0.8 with $\gamma = 5/3$. The spacetime diagram is shown on Figure 7.

The time measurement on the rocket (S') was 3 km. This is the proper time or "eigenzeit" because the clock on the rocket was fixed to a specific location. The length of the track observed from the rocket was the contracted length $4/\gamma = 2.4$ km. The relative velocity observed on the rocket was $v = 2.4/3 = 0.8$, the same value observed on the ground. The proper time of $\Delta t' = 3$ km measured on the rocket was dilated to $\Delta t = 3\gamma = 5$ km on the ground. This is a timelike event with a spacetime interval of $\sqrt{5^2 - 4^2} = \sqrt{3^2} = 3$ km.

18.3. **Trip to Proxima Centauri.** A spaceship travels from Earth to Proxima Centauri, a distance of 4.24 light years at (0.95)c in $(4.24)/(0.95) = 4.463$ yr. The Lorentz transform is:

$$t' = \gamma(t - \frac{v}{c^2}x) = (3.202)[4.463 - (0.95)(4.24)] = 1.393 \text{ yr}$$

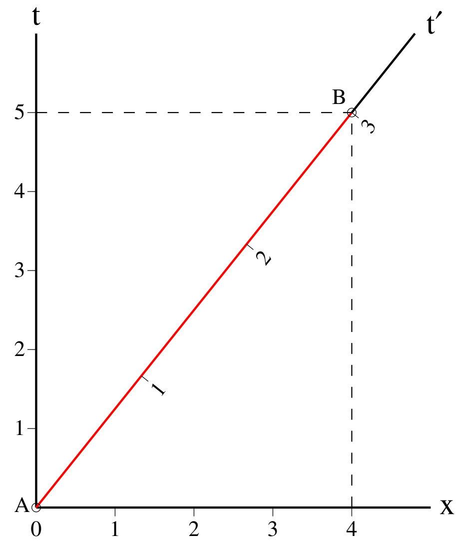

FIGURE 7. Spacetime diagram for rocket speed calibration. Red line is rocket trajectory, from event A at the starting line of the track to event B at the finish line of the 4 km track. The earth coordinate system, using synchronized clocks at A and B, registered a time of 5 km for a velocity of $0.8c$. The proper time registered on the rocket was 3 km.

with $x' = 0$. The time dilation is $\gamma t' = (3.202)(1.393) = 4.463$ yr. The Lorentz contraction of Δx is $\Delta x/\gamma = 4.24/3.202 = 1.324$ light years. The relative velocity is:

$$v = \frac{4.24}{4.463} = \frac{1.324}{1.393} = 0.95$$

The spacetime interval is:

$$\sqrt{4.463^2 - 4.24^2} = \sqrt{1.393^2} = 1.393 \text{ yr}$$

18.4. **Muon decay.** The half-life of a muon is 2.2 μs or 660 m. What is the laboratory half-time for muons traveling at 0.6c? The muon frame provides the "eigenzeit"; think of the half-time as one tick of a muon clock, with $x' = 0$ and $t' = 660$ m. Using the Lorentz transforms with $\gamma = 1.25$ for $v = 0.6$ c:

$$x = \gamma(x' + vt') = \gamma v t' = (1.25)(0.6)(660) = 495 \text{ m}$$
$$t = \gamma(t' + vx') = \gamma t' = (1.25)(660) = 825 \text{ m}$$

The muon proper time of 660 m is dilated to 825 m in the lab frame. The muon velocity in the lab frame is $495/825 = 0.6$.

18.5. **Muon decay on Mt. Washington.** The flux of muons traveling at $0.9938\,c$ was measured at the top of Mt. Washington (height 1920 m) and compared to the flux at the foot of the mountain (actually at MIT). At this speed $\gamma = 9$. Since the half-life is 660 m, the trip of 1920 m corresponds to almost 3 half-lives and the survival rate would be about 1/8 if relativistic effects are ignored.

The Lorentz contraction factor of 9 reduces the distance traveled in the muon frame to $1920/9 = 213$ m. At $v = 0.9938\,c$, the muon time is $d/v = 213/0.9938 = 215$ m. This is $215/660 = 0.326$ half-lives. The survival rate at ground level is

$$\frac{I}{I_\circ} = 2^{-0.326} = 0.8$$

or 80%.

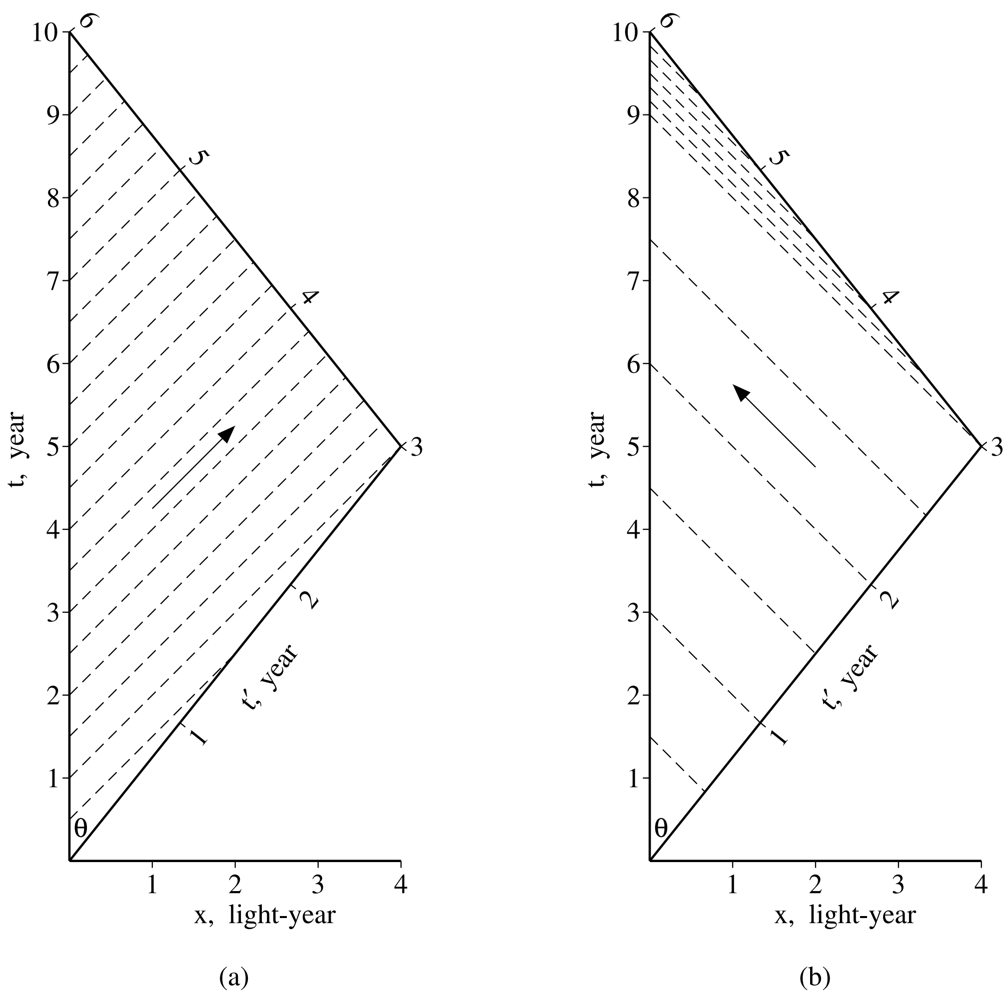

FIGURE 8. Minkowski diagrams for twin paradox. (a): 20 light signals from earth to astronaut; (b)12 light signals from astronaut to earth.

18.6. **Twin paradox.** One member of a pair of twins stays home while the other travels on a spaceship at a velocity of $0.8\,c$ to a star located 4 light-years away. After arrival at the destination, the spaceship reverses direction and returns to earth. During the round trip, 10 years have elapsed on earth but the astronaut twin has aged only 6 years. The time dilation factor is $10/6 = 5/3$. Special relativity predicts that the elapsed time on a clock depends upon its path through spacetime. The paradoxical aspect is the question about which twin is older and which twin is younger when they meet again. The spaceship left the earth and completed a round trip that lasted 10 years. What is wrong with the argument that the earth traveled away from the rocket and returned to the rocket 10 years later, so that the twin on the rocket aged 10 years while the stay-at-home aged only 6 years, since all motion is relative?

The Minkowski diagram is shown on Figure 8. (t, x) coordinates represent the inertial frame of the earth. The t-coordinate is the world line of the earth and the t'-coordinate is the world line of the spaceship. Here, $c = 1$ and the dimensionless relative velocity of the spaceship with respect to the earth is $v = 0.8$. The angle $\theta = \tan^{-1} v = 38.66°$. The scale factor of the t'-axis relative to the t-axis is $1/\sqrt{\cos 2\theta} = 2.134$. The spaceship shifts to a new inertial frame and a new world line at $t' = 3$ yr. The astronaut version of the journey is to travel to the destination in 3 years at $v = 0.8\,c$, turn around, and take 3 more years to return to earth. For the astronaut, the distance to the star is reduced by the Lorentz factor of $\gamma = 5/3$ to 2.4 light-years. The twins agree upon their relative speed of $v = 0.8$. The stay-at-home version of the trip is that the spaceship travels a distance of 4 light-years out and 4 light-years back in 10 years.

Although their clocks run at different rates, the twins must agree upon the Doppler redshift of light signals. The redshift z is defined by

$$z = \frac{\lambda_{\text{obs}} - \lambda_{\text{em}}}{\lambda_{\text{em}}}$$

where $\lambda =$ wavelength. For relativity, it is convenient to work with the redshift factor $(1 + z)$:

$$(1 + z) = \frac{\lambda_{\text{obs}}}{\lambda_{\text{em}}} = \frac{\nu_{\text{em}}}{\nu_{\text{obs}}}$$

ν is the frequency of the light signals. The blueshift factor is the reciprocal of the redshift factor:

$$\frac{1}{(1 + z)} = \frac{\nu_{\text{obs}}}{\nu_{\text{em}}}$$

For velocity v away from the observer, the theoretical relativistic redshift factor is

$$(1 + z) = \sqrt{\frac{1 + v}{1 - v}}$$

For $v = 0.8$, the redshift factor is $\sqrt{(1 + 0.8)/(1 - 0.8)} = 3$.

The exchange of light signals between the astronaut and the earth is shown on Figure 8. The twins agree to send light signals every six months, a frequency of 2 yr^{-1}. Figure 8(a) shows the stay-at-home emitting a total of 20 light signals, all of which were received by the astronaut. The signals were received at a frequency of $2/3$ yr^{-1} during the first three years of astronaut time for a redshift factor of $\nu_{\text{em}}/\nu_{\text{obs}} = (2)/(2/3) = 3$. During the last three years of astronaut time, the redshift factor is $2/6 = 1/3$, or a blueshift factor of 3. Figure 8(b) shows the astronaut emitting a total of 12 light signals, all of which are received by the earth. These signals were received on earth at a frequency of $2/3$ yr^{-1} during the first nine years of earth time for a redshift factor of $\nu_{\text{em}}/\nu_{\text{obs}} = 2/(2/3) = 3$. During the last year of earth time, the redshift factor is $2/6 = 1/3$, or a blueshift factor of 3.

The twins agree upon the redshift of light signals as the rocket travels to its destination, and the blueshift as the rocket returns. Think of the light signals as the ticking of a clock. The twins agreed to send signals twice yearly according to their own clock. The astronaut got 12 ticks but the stay-at-home got 20 ticks, a time dilation factor of $20/12 = 5/3$. This is in agreement with theoretical time dilation factor of $\gamma = 1/\sqrt{1-v^2} = 1/\sqrt{1-0.6^2} = 5/3$.

The change from redshift to blueshift signals the turnaround of the spaceship. As shown of Figure 8(b), this change occurs at 3 years of spacetime time, at the moment that the astronaut feels the deceleration and acceleration of turning around. Figure 8(a) shows that the stay-at-home experiences no acceleration at an earth time of 5 years, as he would if the earth had reversed its direction. The earth stays in the same inertial frame during the round trip. This asymmetry resolves the so-called twin paradox. It is the astronaut who turns arounds, as verified on Figure 8 by the light signals.

In summary, there are two versions of this voyage. In one version, the astronaut travels a distance of 2.4 light-years in 3 years at a speed of 0.8 . After turning around, the spaceship returns to earth at the same speed for a voyage lasting a total of 6 years. The earth version is that the spaceship travels to a star 4 light-years away in 5 years at a speed of 0.8 and then returns at the same speed for a voyage lasting a total of 10 years. For the first 9 years, there is no evidence on earth that the spaceship is returning. At the end of the 9th year, the light signals received on earth change from redshift to blueshift. The astronaut experiences 6 years of heartbeats and the stay-at-home twin gets 10 years of heartbeats. Which of these versions is correct? They are both correct and perfectly consistent if you accept the principles of special relativity about time dilation and length contraction.

An aspect of this voyage that deserves comment is the assumption of instantaneous acceleration and deceleration of the spaceship for a constant speed of $v = 0.8$ in both directions. This velocity could be achieved with an acceleration at $10g$ for 30 days but the distance covered during the accelerations and decelerations would be negligible compared to the distance of 4 light-years to the star. Taking acceleration into account would provide an increase in accuracy at the expense of making the calculation needlessly complex. Probably the actual voyage will be accomplished by accelerating at $1\,g$ for the first half of the outgoing voyage to the star, and then decelerating at $1\,g$ for the second half of the the outgoing voyage. The reverse flight plan would be symmetrical for the spaceship.

18.7. **Relativistic acceleration.** In spite of the principle of equivalence between gravity and acceleration in general relativity, special relativity applies to reference frames accelerated in the absence of gravity forces.

The 4-velocity (u^μ) of a rocket moving at constant velocity $v = dx/dt$ along the x-axis with respect to earth ccordinates (t, x) is defined

$$u^\mu = \left(\frac{dt}{d\tau}, \frac{dx}{d\tau}\right)$$

where τ is proper time measured in the rocket. The 4-velocity has the scalar value

$$u^\mu u_\mu = -1$$

The solution of these linear differential equations, with a suitable constant of integration, is

$$x = \gamma v \tau$$
$$t = \gamma \tau$$

where $\gamma = 1/\sqrt{1 - v^2}$. This simple derivation for the world line of a rocket traveling at constant velocity is intended as preparation for calculating the world line of an accelerated rocket.

The 4-acceleration

$$\mathbf{a} = \frac{d\mathbf{v}}{d\tau}$$

is orthogonal to the 4-velocity:

$$\mathbf{u} \cdot \mathbf{a} = \frac{d}{d\tau}\left(\frac{1}{2}\mathbf{u} \cdot \mathbf{u}\right) = \frac{d}{d\tau}\left(-\frac{1}{2}\right) = 0$$

Using the notation

$$u^0 = \frac{dt}{d\tau}, \quad u^1 = \frac{dx}{d\tau}$$
$$a^0 = \frac{du^0}{d\tau}, \quad a^1 = \frac{du^1}{d\tau}$$

we have for accelerated motion the equations

$$u^\mu u_\mu = -1 \tag{33}$$
$$u^\mu a_\mu = -u^0 a^0 + u^1 a^1 = 0 \tag{34}$$
$$a^\mu a_\mu = g^2 \tag{35}$$

Eq. (33) is the scalar value for the 4-velocity, Eq. (34) is the orthogonality condition for acceleration and velocity, and Eq. (35) is the scalar value of the 4-acceleration.

Consider a co-moving inertial frame traveling at the instantaneous velocity of the rocket. The instantaneous velocity of the rocket relative to the co-moving frame $u^1 = 0$, and from Eq. (34) it follows that $a^0 = 0$ because u^0 is non-zero. If g is the acceleration of the rocket relative to the co-moving frame, then the scalar value of Eq. (35) follows from

$$a^\mu a_\mu = -(a^0)^2 + (a^1)^2 = 0 + g^2 = g^2$$

Integration of the second order differential equations (33)–(35) gives (as before with a suitably adjusted origin):

$$t = g^{-1} \sinh(g\tau) \tag{36}$$
$$x = g^{-1} \cosh(g\tau) \tag{37}$$

We see that

$$x^2 - t^2 = \frac{1}{g^2}$$

so the path of the accelerated rocket is a rectangular hyperbola in a spacetime diagram with earth coordinates (t, x). Also

$$u^0 = \cosh(g\tau) = \gamma, \quad a^0 = g\sinh(g\tau)$$
$$u^1 = \sinh(g\tau) = v\gamma, \quad a^1 = g\cosh(g\tau)$$

and these equations for the components of the 4-velocity and 4-acceleration satisfy Eqs. (33)–(35). The instantaneous velocity of the rocket is

$$v = \tanh(g\tau)$$

Setting g equal to the gravity of the earth:

$$g = g_e = \left[\frac{9.806\,\mathrm{m}}{\mathrm{s}^2}\right]\left[\frac{3.156\times10^7\,\mathrm{s}}{\mathrm{year}}\right]\left[\frac{\mathrm{s}}{2.9979\times10^8\ \mathrm{m}}\right] = 1.032\ \mathrm{year}^{-1}$$

Consider a flight from the Sun to its nearest star, Proxima Centauri, a distance of 4.24 light-years. For the first half of the flight accelerate at g_e over a distance of 2.12 light-years, then for the second half decelerate over the same distance at g_e. Set the origin at $x = t = \tau = 0$. This initial condition is satisfied by Eq. (36) but reset the origin of Eq. (37) with

$$xg = \cosh(g\tau) - 1$$

For the acceleration half, $t = 2.933\,\mathrm{year}$, $\tau = 1.770\,\mathrm{year}$, and the maximum velocity at $t = 2.933\,\mathrm{year}$ is $v = 0.949$. τ and t are unchanged for the deceleration half, so the total distance of 4.24 light-years is covered in an earth time of 5.87 year and a rocket time of 3.54 year.

Flights at constant acceleration can travel enormous distances within a human lifespan. Consider a trip from our Milky Way galaxy to Andromeda, the nearest major galaxy, 2.54 million light-years away. As before, accelerate at g_e over a distance of 1.27 million light years. For the acceleration half, $t = 1.27\times10^6$ year and $\tau = 14.32$ year. The total distance of 2.54 million light-years is covered in an earth time of 2.54 million years, that is, at the speed of light to several significant figures. The rocket time is 28.6 year, well within a human lifespan. Theoretically, it is possible for a human being to leave the Earth and return millions of years later.

18.8. **Derivation of Einstein's equation with gedanken experiment.** Consider two objects of mass m_1 and m_2 at rest and isolated in space. The pair of objects is separated by a distance L at $t = 0$, at which time a pulse of light is emitted in the positive x-direction from object 1 toward object 2. The impulse of the light imparts momentum on object 1 in the negative x direction. At time $t = L/c$, the light strikes object 2 and imparts momentum on it in the positive x direction. Since energy and mass are equivalent, the net effect of the departure of energy from object 1 and arrival of the same energy to object 2 is a transfer of mass from m_1 to m_2.

The relative motion of the two objects is counter to the mutual attractive force of gravity but the duration of the experiment is about 10 orders of magnitude less than the time required for the two objects to come together by gravity forces. Since the velocities of objects 1 and 2 are non-relativistic, absolute space coordinates are used to describe their locations.

At $t = 0$, object 1 emits energy E and suffers a mass change to m_1'. The momentum of a photon is given by $E = cp$ so object 1 acquires a velocity v_1:

$$v_1 = -\frac{E}{cm_1'}$$

The location of object 1 at any time is given by:

$$x_1(t) = -\frac{Et}{cm_1'}$$

When the energy arrives at object 2 at $t = L/c$ it causes a recoil and a change of mass so that:

$$x_2(t) = L + \frac{E}{cm_2'}\left(t - \frac{L}{c}\right)$$

Let $M = m_1 + m_2$. The mass lost by object 1 is equal to the mass gained by object 2, so M is unchanged by the energy transfer. Let the location of the center of mass be $\overline{x}$ before the radiation was emitted from object 1 and $\overline{x}'$ after it was absorbed by object 2:

$$M\overline{x} = (0)m_1 + Lm_2$$

$$M\overline{x}' = m_1'\left[-\frac{Et}{cm_1'}\right] + m_2'\left[L + \frac{E}{cm_2'}\left(t - \frac{L}{c}\right)\right]$$

Assuming that the center of mass of an isolated system does not move, $\overline{x} = \overline{x}'$ and

$$Lm_2 = Lm_2' - \frac{EL}{c^2}$$

$$m_2' - m_2 = \Delta m_2 = \frac{E}{c^2}$$

The energy absorbed by object 2 is associated with an increase in its mass according to $E = mc^2$.

The force of gravity between the two objects is neglected because the time required for gravity to bring the two bodies into contact is ten orders of magnitude greater than the time required for the exchange of a photon.

It is surprising that the relativistic equation $E = mc^2$ is consistent with Newtonian physics. However, we cheated by using the relativistic equation $E = cp$ for a photon.

[home page, Alan L. Myers]