Subjective Theory of Probability

Dutch Book (de Finetti's) Theorem

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Introduction

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- de Finetti theorem provides an alternative view that does not depend on a preliminary concept of independence, and which concentrates attention on the *linearity properties* of expectations.

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- By fair I mean that you should be prepared to accept a payment $p(X)$ from me now in return for giving me an amount X later.
- Your return: $X'(\omega) = X(\omega) - p(X)$. We call this *fair return*.

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These two conditions imply that imply that the collection of all fair returns is a vector space over field $\mathbb{R}$.

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- There is no fair return X' for which $X'(\omega) < 0$ for all $\omega \in \Omega$, with strict inequality for at least one ω.

Properties of fair bets!

Lemma

The previous properties imply that $p(.)$ is a linear, increasing functional on random variables.

Proof.

For constants α and β and random variables X and Y with fair prices $p(X)$ and $p(Y)$, consider the combined effect of the following fair bets:

- You pay me $\alpha p(X)$ to receive αX.
- You pay me $\beta p(Y)$ to receive βY.
- I pay you $p(\alpha X + \beta Y)$ to receive $\alpha X + \beta Y$.

Your net return is $c = p(\alpha X + \beta Y) - \alpha p(X) - \beta p(Y)$.
If $c > 0$, (iii) is violated. If $c < 0$, consider the other side bet to violate (iii). This proves linearity. □

Proof.

To prove that $p(.)$ is increasing, suppose $\forall \omega \in \Omega : \; X(\omega) \geq Y(\omega)$.
If you claim that $p(X) < p(Y)$ then I would be happy for you to accept the bet that delivers $(Y - p(Y)) - (X - p(X)) = -(X - Y) - (p(Y) - p(X))$, which is always < 0. □

Note

If both X and $X - p(X)$ are fair, so is $X - \big(X - p(X)\big)$ with constant return. This imples that $p(X) = 0$.

de Finettin Theorem

Theorem

$p(F_X \cup F_Y) = p(F_X) + p(F_Y)$ for disjoint $F_X, F_Y \subseteq \Omega$. Here we have used the de Finetti notation $p(A) = p(\mathbf{1}_A)$ for $A \subseteq \Omega$.

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As a special case, consider the bet that returns 1 if an event F occurs, and 0 otherwise. The previous theorem implies

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We can similiary show that $p(\Omega) = 1$ and $p(\emptyset) = 0$.

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- The indicator function F ensures that money changes hands only when F occurs.

Theorem

If Ω is partitioned into disjoint events $F_1, \ldots, F_k$. and X is a random variable, then $p(X) = \sum_{i=1}^{k} p(Fi)\, p(X|F_i)$.

Proof.

For a single F_i, argue by linearity that

$$0 = p(XF_i - p(X|F_i)F_i) = p(XF_i) - p(X|F_i)\, p(Fi).$$

Sum over i, using linearity again, together with the fact that $X = \sum_i XFi$, to deduce that $p(X) = \sum_i p(XF_i) = \sum_i p(F_i)p(X|F_i)$, as asserted. □

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$$1 = p(1) = p(\sum_{0\leq t\leq 1} X_t) \stackrel{?}{=} \sum_{0\leq t\leq 1} p(X_t) = \begin{cases} 0 & c = 0 \\ \infty & \text{else} \end{cases}$$