a repertoire of standard problems under one's belt. Since I do not want to talk about the teaching of combinatorics as a whole, I will stop here and get in business.

1 The bijection principle

Just to be formal, I will now state the BP.

The bijection principle (BP) *If there is a bijection between two sets then they have the same number of elements.*

I always illustrate the BP by putting the fingers of my hands tip to tip and asking what is the meaning of this gesture, to which my students almost invariably answer *You have five fingers in each hand.* I expect this wrong answer[4], which gives me the opportunity to clarify things and easily illustrate non-injective and non-surjective functions. Of course there is always a student who makes the (also expected) silly joke about Brazil's president missing a finger in one of his hands, but I quickly restore order and move ahead.

The BP is so much in the nature of a truism that the students' reaction is an immediate "So what?". The answer to this natural question is the philosophy behind the BP, which can be stated as follows. One is faced with the task of counting a set A, but for whatever reason this is difficult. "Luckily" (see below), it is possible to establish a bijection between A and another set B, and we know how to count B. Then A is automatically counted and we are done.

So far so obvious, so what is all the fuss about? It is now that one talks about the hard facts of life. First, it should be stressed that the BP is *not* a counting tool; as it is clear from its statement, it only changes the problem of counting A into the one of counting B. Secondly, finding B and a bijection $f : A \to B$ may be (and most of the time, is) a far from easy task. More than that, one does not find B first and f afterwards, or vice-versa; there is a symbiotic relationship between them and they are, of necessity, created simultaneously. And then there is the problem of proving that f is indeed a bijection.

[4]The right answer, of course, is that the right and left hands have the same number of fingers.

I usually begin to address these points with the following example, copied more or less *verbatim* from [1].

Example 1 *1500 teams compete in the soccer tournament of the Andromeda galaxy. The organizers let the teams know that every game must have a winner and that the team that loses a game is immediately excluded from the tournament. How many games will be played till the champion is known?*

Usually the students will ask about pairing rules, on the basis of reasonings such as "In the first round there will be 750 games, in the second 375 and then how does one make pairings for an odd number of teams?". There is general disbelief when they are told that this is irrelevant. Eventually, if need be with some strong hints, the idea of bijecting the set of games with the set of losers by *game* $\leftrightarrow$ *loser* eventually comes out, giving 1499 as the answer[5]. It takes some doing to convince the students that we have indeed a bijection, but eventually they yield, albeit with some suspicion; after all, this is supposed to be a difficult problem and it is unfair to solve it in such a simple way. But after a while the idea sinks in, and it should; this is indeed a striking application of the BP, in particular because of the utmost simplicity in counting the set of losers and the fact that, indeed, pairing rules are completely irrelevant.

Let's fix some terminology now. One wants to count a set of *objects* A by establishing a bijection f between A and a set of *names* B[6]; in order to count A one is then reduced to *counting names.* I also use *to every object corresponds a unique name* to mean *well defined* and after that *every name comes from a unique object* gives us *bijective* right away. This helps to keep the discussion about the bijectivity of f on a coloquial level, free of technicalities. As a matter of fact, when one gets a good choice of $f : A \to B$, the bijectivity is almost always self-evident and deserves only a cursory mention; it is usually besides the point and distracting to ask for formal proofs.

One now is led to the crux of the matter, which is the choice of names.

[5] An added bonus here is that this reasoning (and, in fact, all others in this article) works for n teams for any n, thereby illustrating another important principle in combinatorics, which I call the *generalization principle.* It can be loosely stated as *good combinatorial reasonings are those which hold when numbers are replaced by letters.* But this is the theme of another article.

[6] One could also use, say, *descriptions* instead of *names* and substitute *naming* by *describing*; other alternatives are possible and reasonable. To each his/her own choice of terminology.

2 The choice of names

What's in a name? For our purposes, a name is a way of describing an object within an agreed upon code of communication, respecting the principles of *precision* and *economy*; of course, names should also be amenable to (hopefully easy) counting. Loosely speaking, in assigning names the following conditions should be (ideally) respected.

First and obviously, the set of objects to be counted should be in one to one correspondence with the set of names. This is what is meant by precision: an object has a unique name and every name uniquely identifies the corresponding object. Secondly, all names should be of the same standard combinatorial type, so that a single counting technique is enough to count them all; I call this *uniformization.* And thirdly, names must be economical, in the sense that they should not contain redundant information. I guess there is no need to justify this requirement by arguments other than aesthetics alone, but if asked to do so I would say that redundant information may (and usually does) obscure the counting process and also conflicts (*ibid*) with uniformization (cf. example 6 below).

Precision and economy are, in a way, opposite to each other. One is always tempted to "make sure" by adding information at the expense of economy, or else to be so economical in giving information that precision is compromised. The (unique) middle point between these concepts is where good names come in. Putting it succintly, good names are those that contain only necessary and sufficient information for recovering their corresponding objects.

In order to make the students add all this to their thinking in a practical way, I ask them find a way to describe to a friend, over the telephone, one or more of the objects to be counted, keeping in mind that telephone rates are very high. This works: if your friend does not know which object you are talking about, you are being imprecise and you have to repeat or to clarify, losing money in the process. Likewise, if you send redundant information, you are wasting time and you lose money too.

Now for some examples. For them I have chosen to present solutions which illustrate the use of the BP; others, of course, are possible [7]. These examples look simple, but I have found all of them to be consistently difficult

[7] I have also chosen to use small numbers for simplicity, but it should be clear that all reasonings work in general .

to beginning students, even for those who are sufficiently conversant with basic principles. After following these examples, the students always get the hang of the "naming" idea, which can then be relegated to the unconscious and called forth when needed.

I note that the pace of the examples below is not the way I work them out in the classroom. There one discusses at lenght, waiting and commenting on the student's ideas and eventually, when things come to a standstill, giving partial ideas so as to let them proceed further. So the solutions I give should be read as rough sketches of discussions to be developed in class.

Example 2 *How many 5-sequences can be made with the digits* $1, 1, 1, 8, 9$*?*

In order to describe such sequences we first discuss economical ways to name them. After a while, the idea of first saying in which position (say, from the left) is the 8 and then do the same for the 9 emerges; for instance, $19118 \leftrightarrow 52$ and $89111 \leftrightarrow 12$. In order to discuss the bijection part, it is enough to ask something like "Which sequence corresponds to 42?" and wait for the right answer. Note that precision and non-redundancy are obvious here, since once we know where the 8 ad the 9 are, there is no need to say anything about the position of the 1's. In this way we establish a bijection between the set of our sequences and the set of 2-sequences of distinct symbols chosen among $\{1, 2, 3, 4, 5\}$; this last is easy to count, and we are done.

Example 3 (circular permutations) In how many ways can 9 people sit around a circular table, all seats being identical?

Here there is always the problem of the meaning of *ways of seating*, which is why this problem is always stated with the no more illuminating restriction of "up to rotation". What I try to do to clarify things is to make the students imagine themselves standing on the center of the table, turning round – always clockwise, say – and pointing out, in order, the the people they see. In this way, all realize that a way of seating can be described by a 9-sequence of people. But then, maybe after a 10-second delay, the whole class protests that a given way of seating can correspond to many such sequences; in other words, they say that an object can have more than one name, which is bad – and one of the main points of this exercise has already been made, with no effort whatsoever. How can we get rid of this bad choice?

Two strategies then offer themselves. The first depends on sticking to what we have and discovering how many possible names a given way of seating can have; I will defer that to the next section. The second does things by a neat trick, which is not so much of a a trick when seen from the point of view of uniformization. One numbers the people from 1 to 9 and then uses only names (i.e., 9-sequences) which begin with 1. Ideally (or with some prodding, if need be) someone will then raise his/her hand and invoke non-redundancy to argue that the beginning 1 is redundant. This finishes the discussion: we are counting permutations of the symbols 2, 3, . . . , 9 and the game is over.

Example 4 *In how many ways can 10 identical candies be distributed among 4 (obviously distinct!) children?*

This is a standard combinatorial problem, which for the beginning student seems to defy solution using only basic tools. And yet, a strikingly beautiful use of the BP makes things surprisingly simple. Let's first order the children and fix this order for the rest of the problem. Say then the children got 3, 4, 1 and 2 candies, in this order; we name this as OOO+OOOO+O+OO. If the children got 4, 1, 0 and 5 candies, we would write OOOO+O++OOOOO. One should now offer a couple of sequences of O's and +'s and ask which candy distributions they came from, which will readily convince the students that we have indeed a bijection. At the end, we are counting the number of 10-sequences made up of ten O's and three +'s, and this is a standard problem, the answer being $\binom{10+3}{3}$.

It should be noted that in this example the names offer themselves to the careful student. It is enough to think of telling a friend about a given candy distribution among the children with as few words as possible; after having agreed on a fixed ordering of the children, one would say "three (pause), four (pause), one (pause), two" which, of course, *is* OOO+OOOO+O+OO.

Example 5 *In how many ways can 10 different rings be distributed among the fingers of one hand? It is assumed that any finger can hold all rings.*

Let's be quick here, since this example is a variation of the previous one. One reads aloud the the 10-sequence of rings, from thumb to little finger and from the base of the fingers to their tips. This is still not a good name, but in the act of reading the jumps between fingers are marked by pauses, and the right names offer themselves again, in this case being 14-sequences of ten distinct symbols (the rings) and four identical pauses. This is standard counting again, the answer being $\frac{14!}{4!}$.