Network Design for Controllability Metrics

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Control performance of a dynamical system can be characterized in terms of its controllability Gramian.

We consider the design of a structural perturbation on the system’s state transition matrix in order to improve the system’s control performance.
We formulate and solve a system design problem.

**Improve worst-case controllability:**

Minimum energy to transfer state in the least favorable direction.

- Minimum energy is proportional to eigenvalue $\lambda_1(W_c^\infty)$
- Worst direction is aligned with eigenvector $v_1(W_c^\infty)$
- Gramian $W_c^\infty$ is given by Lyapunov Equation
- A **bilinear** problem, addressed by a sequence of convex programs

We illustrate the problem on an example multi-agent network.
1 Introduction

2 Problem Formulation

3 Solution to $\mathcal{P}_1$ (worst-case controllability)

4 Numerical Example
Consider the system (network) dynamics described by
\[ \dot{x}(t) = A(G)x(t) + Bu(t), \]  
(1)
where \( x(t) \in \mathbb{R}^n \) denotes the state, and \( u(t) \in \mathbb{R}^p \) is the input signal.

The dynamics \( A(G) \in \mathbb{R}^{n \times n} \) is induced by a directed interdependency graph \( G = (V, E) \) given by a set of nodes \( V \) and a set of edges \( E \).

The input matrix \( B \in \mathbb{R}^{n \times m} \) is such that \([B]_{ik} \neq 0\) if the external input signal \( k \) is available to state \( i \), and \([B]_{ik} = 0\) otherwise.
Consider an input control law $u(t)$ for $t \in [0, t_f]$. Also, let $x_0 = 0$ be the initial state and $x_f$ be the state at the final time.

It is known that the **minimum control energy** to steer the system to $x_f$ is

$$\int_0^{t_f} \|u(\tau)\|^2 d\tau = x_f^T \left[ W_{\infty}^{t_f}(\mathcal{G}) \right]^{-1} x_f,$$

where $W_{\infty}^{t_f}(\mathcal{G})$. The ‘infinite’ controllability Gramian

$$W_{\infty} \equiv W_{\infty}(\mathcal{G}) = \int_0^{\infty} e^{A(\mathcal{G}) \tau} BB^T e^{A(\mathcal{G})^T \tau} d\tau$$

is positive definite if and only if $(A(\mathcal{G}), B)$ is controllable.

Then, $W_{\infty}$ can be computed as the unique solution to:

$$A(\mathcal{G}) W_{\infty} + W_{\infty} A(\mathcal{G})^T + BB^T = 0.$$
Worst-case direction

Because $W_c^\infty$ is positive definite when $(A(G), B)$ is controllable, it can be described as

$$W_c^\infty = V \text{diag}(\lambda_1, \ldots, \lambda_n) V^T, \quad V = [v_1 | \ldots | v_n],$$

where $\{(\lambda_i, v_i)\}_{i=1}^n$ are eigenvalue-eigenvector pairs associated with $W_c^\infty$.

We assume that $0 < \lambda_1 \leq \ldots \leq \lambda_n$, where $\lambda_{\min} = \lambda_1$ and $\lambda_{\max} = \lambda_n$.

Therefore, it follows that the total energy incurred by the minimum energy control in a specific final state $x_f = cv_i$ is $c^2 \lambda_i^{-1}$.

In the worst case, the most energy-consuming states are those in the direction of $v_1$, i.e., the eigenvector associated with $\lambda_{\min}(W_c^\infty(G))$. 
We propose a scenario where we **re-design** the corresponding dynamics, while **satisfying the interdependency graph constraints**.

Subsequently, equation (1) becomes as follows:

\[
\dot{x}(t) = [A(G) + \Delta(G)]x(t) + Bu(t),
\]

(2)

where \([\Delta(G)]_{ij} \in [\nu_{ij}, \mu_{ij}] \subset \mathbb{R}\) for \((j, i) \in \mathcal{E}\), and \([\Delta(G)]_{ij} = 0\) otherwise.

Simply speaking, we perform a **finite additive structural perturbation** on the dynamics to ensure desirable control properties measured by spectral properties of the ‘infinite’ controllability Gramian.
### $P_1$ (Worst-case controllability)

Given the interdependency graph $G$ and $(A(G), B)$ controllable, find $\Delta(G)$, with $[\Delta(G)]_{ij} \in [\nu_{ij}, \mu_{ij}] \subseteq \mathbb{R}$ for $(j, i) \in E$ and $[\Delta(G)]_{ij} = 0$ otherwise, such that $(A(G) + \Delta(G), B)$ is controllable and

$$\max_{\Delta(G), W_c^\infty \in \mathbb{S}_+} \lambda_{\text{min}}(W_c^\infty)$$

subject to

$$(A(G) + \Delta(G))W_c^\infty + W_c^\infty(A(G) + \Delta(G))^\top + BB^\top = 0.$$
1 Introduction

2 Problem Formulation

3 Solution to \( P_1 \) (worst-case controllability)

4 Numerical Example
Let $A \equiv A(G)$ and notice that the structural perturbation $\Delta \equiv \Delta(G)$ and the infinite controllability Gramian $W_c^\infty \in \mathbb{S}^n_+$ are related by

$$(A + \Delta)W_c^\infty + W_c^\infty(A + \Delta)^T + BB^T = 0,$$

which involves a sum of bilinear terms in $\Delta$ and $W_c^\infty$.

In particular, let the matrices $M \in \mathbb{R}^{n \times 2n}$, $N \in \mathbb{R}^{2n \times n}$, and $Q \in \mathbb{R}^{n \times n}$ be such that $Q := -BB^T$,

$$M \equiv M(\Delta, W_c^\infty) := \begin{bmatrix} A + \Delta & W_c^\infty \end{bmatrix}, \text{ and}$$
$$N \equiv N(\Delta, W_c^\infty) := \begin{bmatrix} W_c^\infty & A + \Delta \end{bmatrix}^T.$$

Thus, we have that (3) can be rewritten as the BME

$$MN = Q,$$

which is also satisfied when

$$QM - N = 0_{n \times n} \iff \text{rank}(Q - MN) = 0.$$
Following a similar strategy to [Doelman and Verhaegen, 2016], we consider the structured matrix \( Z \in \mathbb{R}^{3n \times 3n} \), defined as

\[
Z = \left[
\begin{array}{cc}
Z_{11} & Z_{12} \\
Z_{21} & Z_{22}
\end{array}
\right] = \left[
\begin{array}{cc}
Q + XY + MY + XN & M + X \\
N + Y & I_{2n}
\end{array}
\right],
\]

which is parameterized by the matrices \( X \in \mathbb{R}^{n \times 2n} \) and \( Y \in \mathbb{R}^{2n \times n} \).

By the Schur complement of \( Z_{22} \), from rank additivity, we have

\[
\text{rank}(Z) = \text{rank}(Z_{22}) + \text{rank}(Z_{11} - Z_{12}Z_{22}^{-1}Z_{21})
= \text{rank}(I_{2n}) + \text{rank}(Z_{11} - (M + X)(N + Y))
= 2n + \text{rank}(Q - MN).
\]

The minimum is when \( \text{rank}(Q - MN) = 0 \), which implies \( MN = Q \).

The matrix \( Z \) has affine dependency on \( \Delta \) and \( W_c^\infty \) (through \( M \) and \( N \)).
Convex Relaxation

Given $X$ and $Y$, the **rank minimization problem** in $Z(\Delta, W_c^\infty; X, Y)$ can be further relaxed to a convex problem.

We consider the **nuclear norm**, denoted by $\|Z(\Delta, W_c^\infty; X, Y)\|_*$, which can be formulated as an SDP [Fazel, 2002], including structural constraints on allowable perturbations.

As shown in [Doelman and Verhaegen, 2016], by relying on sequential convex programming, one can produce a converging sequence

$$\{\Delta^{(k)}, W_c^\infty(k)\}_k$$

(or, equivalently,$$\{M^{(k)}, N^{(k)}\}_k$$),

parametrized by $\{X^{(k)}, Y^{(k)}\}_k$, that will satisfy

$$(A + \Delta^{(k)})W_c^\infty(k) + W_c^\infty(k)(A + \Delta^{(k)})^T + BB^T = 0,$$

if $\text{rank} \left[ Z(\Delta^{(k)}, W_c^\infty(k); X^{(k)}, Y^{(k)}) \right] = 2n.$
Perturbation and Target Controllability Constraints

The convex formulation allows semidefinite and affine constraints in $\mathcal{P}_1$.

The affine constraints encapsulate allowable structural perturbations
\[
\nu_{ij} \leq [\Delta]_{ij} \leq \mu_{ij}, \quad (j, i) \in \mathcal{E}
\]
\[
[\Delta]_{ij} = 0, \quad (j, i) \in \mathcal{E}^c.
\]

The controllability objective in $\mathcal{P}_1$ can be written as an SDP in epigraph form
\[
\max_{W_c^\infty \in \mathbb{S}_+^n} \lambda_{\min}(W_c^\infty) \iff \max_{\delta \in \mathbb{R}, W_c^\infty \in \mathbb{S}_+^n} \delta
\]
\[
s.t. \quad W_c^\infty - \delta I_n \succeq 0.
\]

We consider a target improvement $\delta = \overline{\lambda}$, which is feasible for $\mathcal{P}_1$ if there exist $\Delta$ and $W_c^\infty$ such that
\[
W_c^\infty - \overline{\lambda} I_n \preceq 0.
\]
Thus, we propose a sequence of convex optimization problems $C_1^{(k)}(X^{(k)}, Y^{(k)}, \bar{\lambda})$ that are described as follows.

$$C_1(X, Y, \bar{\lambda})$$

Given a pair $(X, Y)$ and a target $\bar{\lambda}$, find a solution to

$$\min_{\Delta \in \mathbb{R}^{n \times n}, W_c^\infty \in \mathbb{S}_+^n} \|Z(\Delta, W_c^\infty; X, Y)\|_* \quad (8)$$

subject to

$$\text{(5), (6), (7)}$$

where the objective seeks to enforce the Lyapunov BME.

Also, following the above reasoning, the following result holds.

**Theorem**

*The solution to $\mathcal{P}_1$ is given by the solution to $C_1(X, Y, \bar{\lambda})$ for the maximum value $\bar{\lambda}$, as well as some $X$ and $Y$, such that $MN = Q$.***
In summary, we propose to solve feasibility problems associated with $P_1$ for increasing values of $\bar{\lambda}$, by invoking Algorithm 1, which consists of solving consecutive convex relaxations $C_1(X, Y, \bar{\lambda})$.

We start by considering the initial points $X^{(1)} = -[A \ W_c^\infty]$, and $Y^{(1)} = -[W_c^\infty \ A]^T$, corresponding to $\Delta = 0$, and $W_c^\infty$ as a solution to $AW_c^\infty + W_c^\infty A^T = -BB^T$.

The numerical stopping condition is given by the relative residual of the bilinear inequality constraint, i.e., $\|M^{(k)}N^{(k)} - Q\|_*/\|Q\|_* < \epsilon \ll 1$.

**Algorithm 1** Feasibility sequence for $P_1$

1. given $X^{(1)}, Y^{(1)}, \bar{\lambda}$
2. while $\|M^{(k)}N^{(k)} - Q\|_*/\|Q\|_* > \epsilon$ do
3. solve $C_1(X^{(k)}, Y^{(k)}, \bar{\lambda})$
4. let $X^{(k+1)} = -M^{(k)}$, $Y^{(k+1)} = -N^{(k)}$
5. end while
1. Introduction

2. Problem Formulation

3. Solution to $P_1$ (worst-case controllability)

4. Numerical Example
Figure: Multi-agent network considered, with agents 2 and 3 selected as leaders.
The example network is described in terms of the system matrix

\[ A = \begin{bmatrix} 
-1.393 & 0.559 & 0 & 0 & 0 \\
0.732 & -0.781 & 0.581 & 0.071 & 0.374 \\
0 & 0.034 & -0.987 & 0.658 & 0 \\
0.575 & 0 & 0.976 & -1.393 & 0 \\
0.442 & 0.778 & 0.569 & 0 & -1.372 
\end{bmatrix}. \]

The input matrix is given by

\[ B = \begin{bmatrix} 
0 & 0 & 1 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 \\
\end{bmatrix}^T. \]

The infinite controllability Gramian described by

<table>
<thead>
<tr>
<th>( \lambda_1 )</th>
<th>( \lambda_2 )</th>
<th>( \lambda_3 )</th>
<th>( \lambda_4 )</th>
<th>( \lambda_5 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>3.08 \times 10^{-5}</td>
<td>0.031</td>
<td>0.131</td>
<td>0.635</td>
<td><strong>633.666</strong></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>( v_1 )</th>
<th>( v_2 )</th>
<th>( v_3 )</th>
<th>( v_4 )</th>
<th>( v_5 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.798</td>
<td>0.442</td>
<td>-0.260</td>
<td>-0.146</td>
<td>0.282</td>
</tr>
<tr>
<td>-0.009</td>
<td>-0.150</td>
<td>0.637</td>
<td>-0.276</td>
<td>0.704</td>
</tr>
<tr>
<td>-0.001</td>
<td>0.322</td>
<td>0.250</td>
<td>0.893</td>
<td>0.192</td>
</tr>
<tr>
<td>0.307</td>
<td>-0.797</td>
<td>-0.320</td>
<td>0.323</td>
<td>0.251</td>
</tr>
<tr>
<td>-0.519</td>
<td>0.209</td>
<td>-0.601</td>
<td>-0.030</td>
<td>0.570</td>
</tr>
</tbody>
</table>

we note the low value of \( \lambda_1 \).
We set target $\bar{\lambda} = 0.1$, and perturbation bounds $\nu_{ij} = -1$, $\mu_{ij} = 1$.

Figure: Sequence of values of the relative residual of the bilinear equality constraint $\|M^{(k)}N^{(k)} - Q\|_*/\|Q\|_*$ (top) and $\log_{10}(\lambda_1^{(k)}/\bar{\lambda})$ (bottom), obtained from Algorithm 1 for problem $\mathcal{P}_1$. 
Resulting Perturbation and Gramian

The solution $\Delta p_1$ obtained is

$$
\Delta p_1 = \begin{bmatrix}
1.00 & -0.400 & 0 & 0 & 0 \\
-1.00 & 0.435 & 0.172 & 0.165 & -0.174 \\
0 & 0 & 0.301 & -0.184 & 0 \\
-0.447 & 0 & 0.040 & 0.469 & 0 \\
-1.00 & 0 & -0.544 & 0 & 0.675
\end{bmatrix}.
$$

with active constraints in bold.

The resulting controllability Gramian is described by

<table>
<thead>
<tr>
<th>$\lambda_1$</th>
<th>$\lambda_2$</th>
<th>$\lambda_3$</th>
<th>$\lambda_4$</th>
<th>$\lambda_5$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.100</td>
<td>0.100</td>
<td>0.308</td>
<td>1.191</td>
<td>633.666</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>v1</th>
<th>v2</th>
<th>v3</th>
<th>v4</th>
<th>v5</th>
</tr>
</thead>
<tbody>
<tr>
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<td>0.660</td>
<td>0.278</td>
<td>0.282</td>
</tr>
<tr>
<td>-0.191</td>
<td>-0.276</td>
<td>-0.626</td>
<td>0.006</td>
<td>0.704</td>
</tr>
<tr>
<td>0.407</td>
<td>-0.365</td>
<td>0.245</td>
<td>-0.777</td>
<td>0.192</td>
</tr>
<tr>
<td>-0.554</td>
<td>0.619</td>
<td>0.175</td>
<td>-0.464</td>
<td>0.251</td>
</tr>
<tr>
<td>0.555</td>
<td>0.426</td>
<td>0.286</td>
<td>0.322</td>
<td>0.570</td>
</tr>
</tbody>
</table>

with a many-fold improvement ($> 10^4$) in worst-case controllability.
Conclusion and Future Work

- We addressed **constrained design of system dynamics** to improve performance as a function of the **controllability Gramian**.

- The **worst-case performance** problem can be cast as an optimization problem with **bilinear matrix equality** constraints.

- We validated our approach in the context of multi-agent networks.
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- We addressed **constrained design of system dynamics** to improve performance as a function of the **controllability Gramian**.

- The **worst-case performance** problem can be cast as an optimization problem with **bilinear matrix equality** constraints.

- We validated our approach in the context of multi-agent networks.

As **future work**, we intend to address the following aspects:

- Other objectives (e.g. trace, sum of $r$ eigenvalues, etc.);
- Discrete-time dynamics;
- Specific optimization, convergence guarantees;
- Large-scale systems;
- Decentralized solution;
- Other application domains.
Becker, C., Pequito, S., Pappas, G., and Preciado, V.  
Network design for controllability metrics.  

Sequential convex relaxation for convex optimization with bilinear matrix equalities.  

*Matrix rank minimization with applications*.  
Thank you.

Any questions?