# Dynamic Programming

CIS 110, Fall 2012

University of Pennsylvania

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Dynamic programming records saves computation for reuse later.

- Programming: in the optimization sense ("Linear Programming")
- Dynamic: "... it's impossible to use [it] in a pejorative way." (Richard Bellman)
- The name was designed to sound cool to RAND management and the US Department of Defense
- A more descriptive term is look-up table

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When is dynamic programming useful?

- Exponential number of solutions
- Cost of soluion is recursively computed
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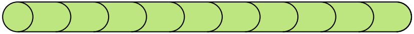
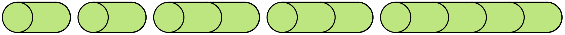
The trick is exposing the common subproblems

# Rod Cutting

Start with a rod of integer length $n$ . . .



. . . and cut it into several smaller pieces (of integer length).



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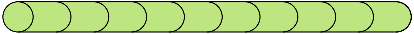

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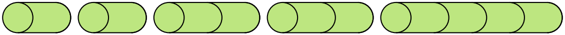

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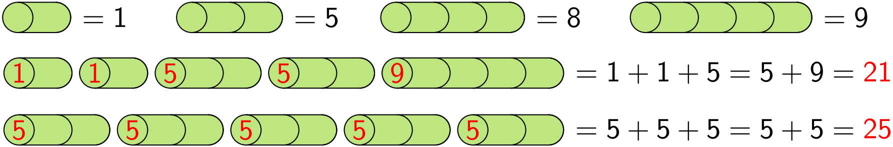

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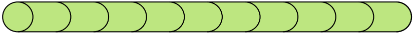

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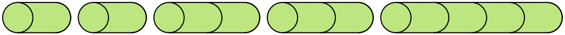

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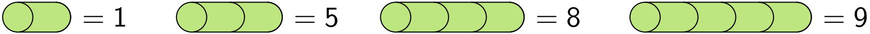
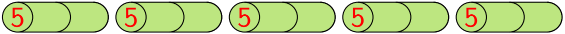




$= 1 + 1 + 5 = 5 + 9 = 21$




$= 5 + 5 + 5 = 5 + 5 = 25$

How should we cut the rod into pieces?

- $2^{n-1}$ possibilities for a rod of length $n$

# Rod Cutting

First rod-cutting strategy (brute-force):

- For every possible cut, compute the value of the left part plus the value of optimally cutting the right part. Take the best cut.

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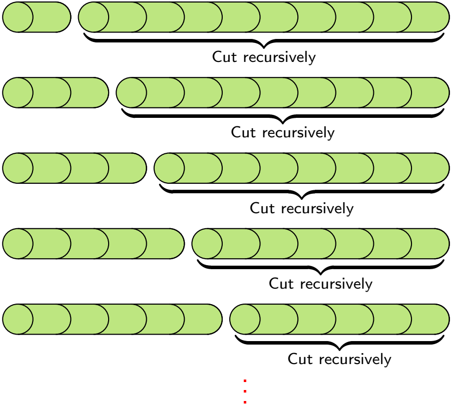

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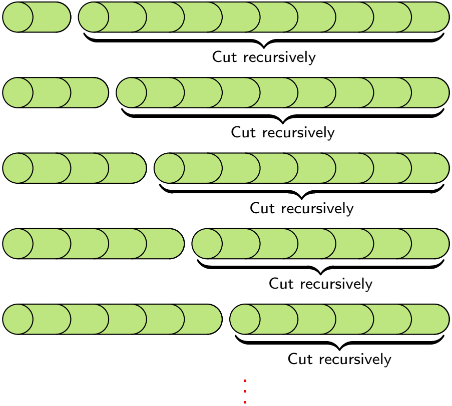


- Exponential number of recursive calls!

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Second rod-cutting strategy (top-down):

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- Find value of optimal cut for right part in table
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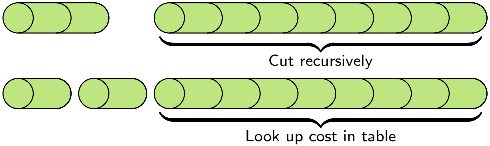

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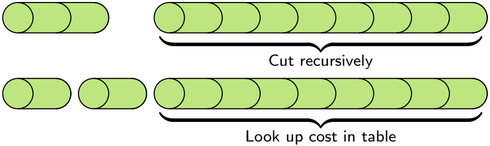



- Reduces computation from $O(2^n)$ to $O(n^2)$ (Why?)
- Requires an array of length $n$ to store intermediate computations

# Rod Cutting

Thrid rod-cutting strategy (bottom-up):

- Compute the value of a rod of length 1. Store it.
- Compute the value of a rod of length 2. You can only cut it into rods of length 1. The value of a rod of length 1 is already computed, so there is no recursion.
- Compute the value of successively longer rods up to length $n$. The optimal values of shorter rods are always computed first so there is no recursion.

# Sequence Matching

- Human genes are coded by four bases: Adenine (A), Thymine (T), Guanine (G), Cytosine (C)
- DNA undergoes mutations with each copy:
  - Substitutions: replace one base with another
  - Deletions: some bases are dropped
- Suppose we isolate a gene in a new organism:

  `A A C A G T T A C C`

  Predict function by comparing to genes in know, organism:

  *e.g.* `T A A G G T C A`
- How similar are `A A C A G T T A C C` and `T A A G G T C A`?

# Sequence Matching

How similar are `A A C A G T T A C C` and `T A A G G T C A`?

- How many mutations to change first sequence into second?
- How (un)likely is each mutation

Edit Distance: minimum cost to convert one string into another.

- Each change (mutation) has an associated cost:

| | |
|---|---|
| Gap | 2 |
| Mismatch | 1 |
| Match | 0 |

- Example matchings:

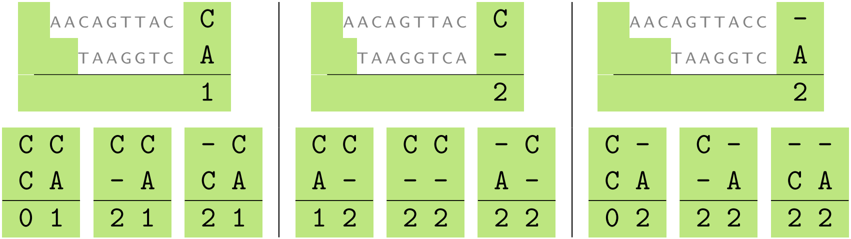
| | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| | A | A | C | A | G | T | T | A | C | C |
| | T | A | A | G | G | T | C | A | - | - |
| **8** | 1 | 0 | 1 | 1 | 0 | 0 | 1 | 0 | 2 | 2 |

| | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| A | A | C | A | G | T | T | A | C | C | |
| T | A | - | A | G | G | T | - | C | A | |
| 1 | 0 | 2 | 0 | 0 | 1 | 0 | 2 | 0 | 1 | **7** |

# Edit Distance

Brute-force recursive solution:

- Start at end of sequence and work backwards




- Recurse until we have all possible matches, then find minimum
- Three recursive calls per node $\Rightarrow$ $O(3^n)$ matching cost
- We need to do better!

# Edit Distance

Dynamic Programming recursive solution

- Consider a pair of characters in the middle:

  `A A C A G T T A C C`

  `T A A G G T C A`

- What is the cost of matching from this pair of Gs to the end?
  - Cost of matching `G`s (0) + *lowest* cost of matching `T T A C C` to `T C A`.
  - Brute force solution computes *all possible* costs
- Idea: For each pair of characters, keep track of best match up to end

# Dynamic Programming

Idea: For each pair of characters, keep track of best match to end

| | A | A | C | A | G | T | T | A | C | C | - |
|---|---|---|---|---|---|---|---|---|---|---|---|
| T | | | | | | | | | | | 16↓ |
| A | | | | | | | | | | | 14↓ |
| A | | | | | | | | | | | 12↓ |
| G | | | | | | | | | | | 10↓ |
| G | | | | | | | | | | | 8↓ |
| T | | | | | | | | | | | 6↓ |
| C | | | | | | | | | | | 4↓ |
| A | | | | | | | | | | | 2↓ |
| - | 20→ | 18→ | 16→ | 14→ | 12→ | 10→ | 8→ | 6→ | 4→ | 2→ | 0 |

Initialization:

- Cost of zero-length match (lower right) is zero
- Inserting a gap (move right or down in table) costs two

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| T | | | | | | | | | | | 6↓ |
| C | | | | | | | | | | | 4↓ |
| A | | | | | | | | | | 1↘ | 2↓ |
| - | 20→ | 18→ | 16→ | 14→ | 12→ | 10→ | 8→ | 6→ | 4→ | 2→ | 0 |

Iteration:

- Work back from lower right
- Cost of cell $(i, j)$ is

$$C(i, j) = \min(C(i + i, j) + 2, C(i, j + 1) + 2, C(i + 1, j + 1)) + \delta$$

where $\delta = 1$ if the $i$'th character of string A and $j$'th character of string B are identical.

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| | A | A | C | A | G | T | T | A | C | C | - |
|---|---|---|---|---|---|---|---|---|---|---|---|
| T | 7↘ | 6↘ | 6↘ | 7↓ | 9↘ | 8↘ | 9↘ | 11↓ | 13↘ | 14↓ | 16↓ |
| A | 8↘ | 6↘ | 5↘ | 5↘ | 7↘ | 8↘ | 8↘ | 9↘ | 11↘ | 12↓ | 14↓ |
| A | 10↘ | 8↘ | 6→ | 4↘ | 5↘ | 6↘ | 7↘ | 7↘ | 9↘ | 10↓ | 12↓ |
| G | 12↘ | 10↘ | 8↘ | 6↘ | 4↘ | 4↘ | 5↘ | 6↘ | 7↘ | 8↓ | 10↓ |
| G | 13→ | 11→ | 9→ | 7→ | 5↘ | 4↘ | 3↘ | 4↘ | 5↘ | 6↓ | 8↓ |
| T | 15↘ | 13↘ | 11→ | 9→ | 7→ | 5↘ | 3↘ | 2↘ | 3↘ | 4↓ | 6↓ |
| C | 16→ | 14→ | 12↘ | 11↘ | 9↘ | 7↘ | 5↘ | 3→ | 1↘ | 2↘ | 4↓ |
| A | 18↘ | 16↘ | 14→ | 12↘ | 10→ | 8→ | 6→ | 4↘ | 3↘ | 1↘ | 2↓ |
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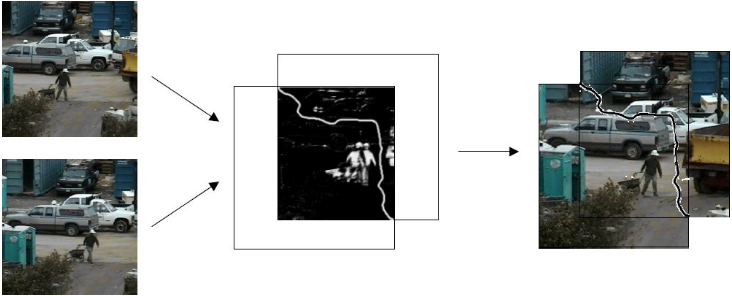
|---|---|---|---|---|---|---|---|---|---|---|---|
| T | 7$\searrow$ | 6$\searrow$ | 6$\searrow$ | 7$\downarrow$ | 9$\searrow$ | 8$\searrow$ | 9$\searrow$ | 11$\downarrow$ | 13$\searrow$ | 14$\downarrow$ | 16$\downarrow$ |
| A | 8$\searrow$ | 6$\searrow$ | 5$\searrow$ | 5$\searrow$ | 7$\searrow$ | 8$\searrow$ | 8$\searrow$ | 9$\searrow$ | 11$\searrow$ | 12$\downarrow$ | 14$\downarrow$ |
| A | 10$\searrow$ | 8$\searrow$ | 6$\rightarrow$ | 4$\searrow$ | 5$\searrow$ | 6$\searrow$ | 7$\searrow$ | 7$\searrow$ | 9$\searrow$ | 10$\downarrow$ | 12$\downarrow$ |
| G | 12$\searrow$ | 10$\searrow$ | 8$\searrow$ | 6$\searrow$ | 4$\searrow$ | 4$\searrow$ | 5$\searrow$ | 6$\searrow$ | 7$\searrow$ | 8$\downarrow$ | 10$\downarrow$ |
| G | 13$\rightarrow$ | 11$\rightarrow$ | 9$\rightarrow$ | 7$\rightarrow$ | 5$\searrow$ | 4$\searrow$ | 3$\searrow$ | 4$\searrow$ | 5$\searrow$ | 6$\downarrow$ | 8$\downarrow$ |
| T | 15$\searrow$ | 13$\searrow$ | 11$\rightarrow$ | 9$\rightarrow$ | 7$\rightarrow$ | 5$\searrow$ | 3$\searrow$ | 2$\searrow$ | 3$\searrow$ | 4$\downarrow$ | 6$\downarrow$ |
| C | 16$\rightarrow$ | 14$\rightarrow$ | 12$\searrow$ | 11$\searrow$ | 9$\searrow$ | 7$\searrow$ | 5$\searrow$ | 3$\rightarrow$ | 1$\searrow$ | 2$\searrow$ | 4$\downarrow$ |
| A | 18$\searrow$ | 16$\searrow$ | 14$\rightarrow$ | 12$\searrow$ | 10$\rightarrow$ | 8$\rightarrow$ | 6$\rightarrow$ | 4$\searrow$ | 3$\searrow$ | 1$\searrow$ | 2$\downarrow$ |
| - | 20$\rightarrow$ | 18$\rightarrow$ | 16$\rightarrow$ | 14$\rightarrow$ | 12$\rightarrow$ | 10$\rightarrow$ | 8$\rightarrow$ | 6$\rightarrow$ | 4$\rightarrow$ | 2$\rightarrow$ | 0 |

Recovering the best alignment:

- Final cost is in cell $(0, 0)$
- Follow arrows to reconstruct string
- $\rightarrow$ aligns letter in the current *column* with a gap
- $\downarrow$ aligns letter in the current *row* with a gap
- $\searrow$ matches letters in current row and column with each other
- Total running time: $O(mn)$!

# Some More Examples

- Image Compositing
  - Given a set of overlapping images, what is the best way to stitch them?
  - Cut the images along an “invisible” seam, and splice them together.
  - The optimal seam can be found through dynamic programming.
    - Even better: shortest path



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# Some More Examples

- Matrix parenthesization
  - Need to multiply a sequence of rectangular matrices
  - Which matrices should be multiplied first to minimize the number of operations

$$\left( \begin{pmatrix} a_1 & a_2 & a_3 \end{pmatrix} \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} \right) \begin{pmatrix} c_1 & c_2 & c_3 \end{pmatrix} = \begin{pmatrix} a_1 b_1 + a2 b_2 + a_3 b_3 \end{pmatrix} \begin{pmatrix} c_1 & c_2 & c_3 \end{pmatrix}$$

$$\begin{pmatrix} a_1 & a_2 & a_3 \end{pmatrix} \left( \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} \begin{pmatrix} c_1 & c_2 & c_3 \end{pmatrix} \right) = \begin{pmatrix} a_1 & a_2 & a_3 \end{pmatrix} \begin{pmatrix} b_1 c_1 & b_1 c_2 & b_1 c_3 \\ b_2 c_1 & b_2 c_2 & b_2 c_3 \\ b_3 c_1 & b_3 c_2 & b_3 c_3 \end{pmatrix}$$

# Some More Examples

- Matrix parenthesization
  - Need to multiply a sequence of rectangular matrices
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- Seam carving (see demo)
  - Shrink an image by finding one row or column of pixels to remove
  - The seam doesn't have to be straight — it can wiggle
  - Use dynamic programming to find the best set of pixels to remove