Programming Languages and Techniques (CIS1200)

Lecture 11

Abstract types: Sets

Chapter 10

Announcements (1)

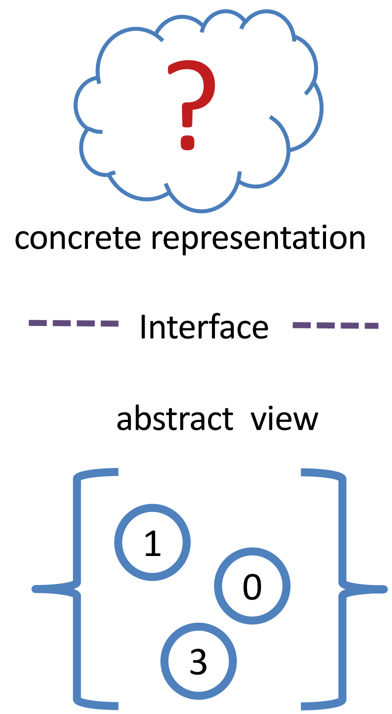
- Homework 3 is due tomorrow at 11.59pm
  - Practice with BSTs, generic functions, first-class functions, and abstract types
- Reading: Chapters 8, 9, and 10 of the lecture notes

Announcements (2)

- Midterm 1: Friday, February 14th
  - Coverage: up to Wednesday, Feb 12th (Chapters 1-10)
  - During lecture
    Last names: A – Z Meyerson Hall B1
  - 60 minutes; closed book, single-sided handwritten letter size notes allowed
  - Review Material
    - old exams on the web site ("schedule" tab)
  - **Review Session**
    - **Wednesday, Feb 12, 7:00-9:00pm, Towne 100** (will be recorded)
    - Review Videos available on canvas

Review: Abstract types (e.g., set)

- An abstract type is defined by its *interface* and its *properties*, not its representation
- Interface: defines operations on the abstract type
  - There is an empty set
  - There is a way to add elements to a set to make a bigger set
  - There is a way to test membership
- Properties: define how the operations interact with each other
  - Elements that were added can be found in the set
  - Adding an element a second time doesn't change the elements of a set
  - Adding in a different order doesn't change the elements of a set
- *Any* concrete type that satisfies the interface and properties can implement a set
- ***Clients of an implementation can only access what is explicitly mentioned in the abstract type's interface***



Set Interface

```
module type SET = sig

   type 'a set

   val empty      : 'a set
   val add        : 'a -> 'a set -> 'a set
   val member     : 'a -> 'a set -> bool
   val equals     : 'a set -> 'a set -> bool

end
```

```
module UnorderedListSet : SET = struct

   type 'a set = 'a list
   …
end
```

```
module OrderedListSet : SET = str

   type 'a set = 'a list
   …
end
```

```
module BSTSet : SET = struct

   type 'a tree =
     | Empty
     | Node of 'a tree * 'a * 'a tree

   type 'a set = 'a tree
   …
end
```

Equality of Sets

When defining an abstract type, you may need to define an *abstract notion of equality*

- The built-in "structural equality" (written =) may not be appropriate for all implementations
- Clients of the abstract type should use the '`equals`' function when comparing sets
- Other generic operations, like < and > may also be affected

Equality of Sets

- The SET interface includes

```
val equals : 'a set -> 'a set -> bool
```

This function should return true when both sets contain same elements

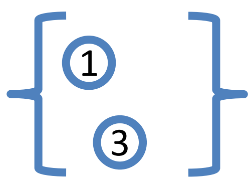
- Can we use OCaml's built-in `=` to compare sets?
  - This generic, built-in equality operation = compares the *structure* of its two inputs to see whether they are the same
- With unordered lists, NO!

These two values *are not* = as lists

`3::1::1::[]`

concrete representation

abstract view

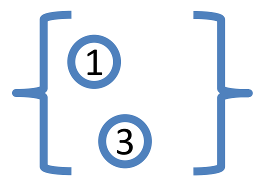


`add 3 (add 1 (add 1 empty))`

`1::3::[]`

concrete representation

abstract view



`add 1 (add 3 empty)`

These two values *are equal* as sets

Equality of Sets

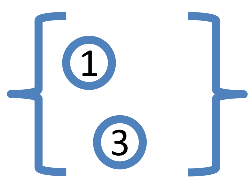
- The SET interface includes

```
val equals : 'a set -> 'a set -> bool
```

This function should return true when both sets contain same elements

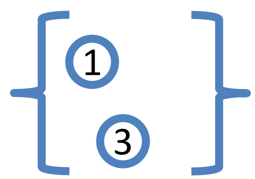
- Can we use OCaml's built-in `=` to compare sets?
  - This generic, built-in equality operation = compares the *structure* of its two inputs to see whether they are the same
- With strictly ordered lists, YES!

`1::3::[]` concrete representation / abstract view



`add 3 (add 1 (add 1 empty))`

`1::3::[]` concrete representation / abstract view



`add 1 (add 3 empty)`

These two values *are =* as lists

These two values *are equal* as sets

Abstract Types

Abstract types: BIG IDEA

Hide the *concrete representation* of a type behind an *abstract interface* to preserve **representation invariants**

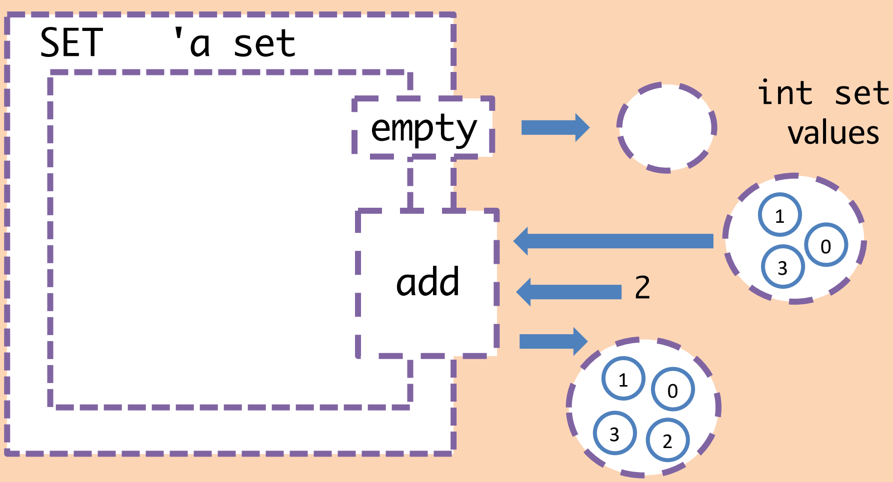
- Example representation invariants
  - Sets implemented as lists, which must be strictly ordered (no duplicates)
  - Sets implemented as binary tree, which must satisfy the BST invariant
- If the set type is abstract, and *all* operations preserve invariants, then invariants **must** hold for *all* sets in the program!
  - Example: if all sets implemented as lists are strictly ordered, then the `=` operation implements set equality
  - Example: if all sets implemented as trees satisfy the BST invariant, then the lookup function can *assume* that its input is a BST

Abstract types: BIG IDEA

Hide the *concrete representation* of a type behind an *abstract interface* to preserve **representation invariants**

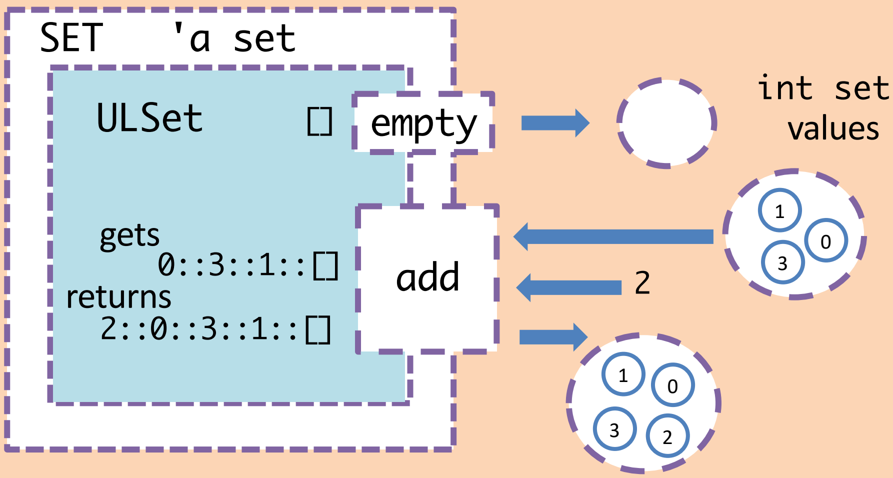
- An abstract interface **restricts** how other parts of the program can interact with the data
  - Type checking ensures that the **only** way to create a set is with the operations in the interface (empty, add, etc.)
  - Type checking ensures that clients cannot depend on whether the sets are implemented as trees or lists
- Benefits
  - **Safety**: The other parts of the program can't violate invariants, which would cause bugs
  - **Modularity**: It is possible to change the implementation without changing the rest of the program

Encapsulation and Modularity



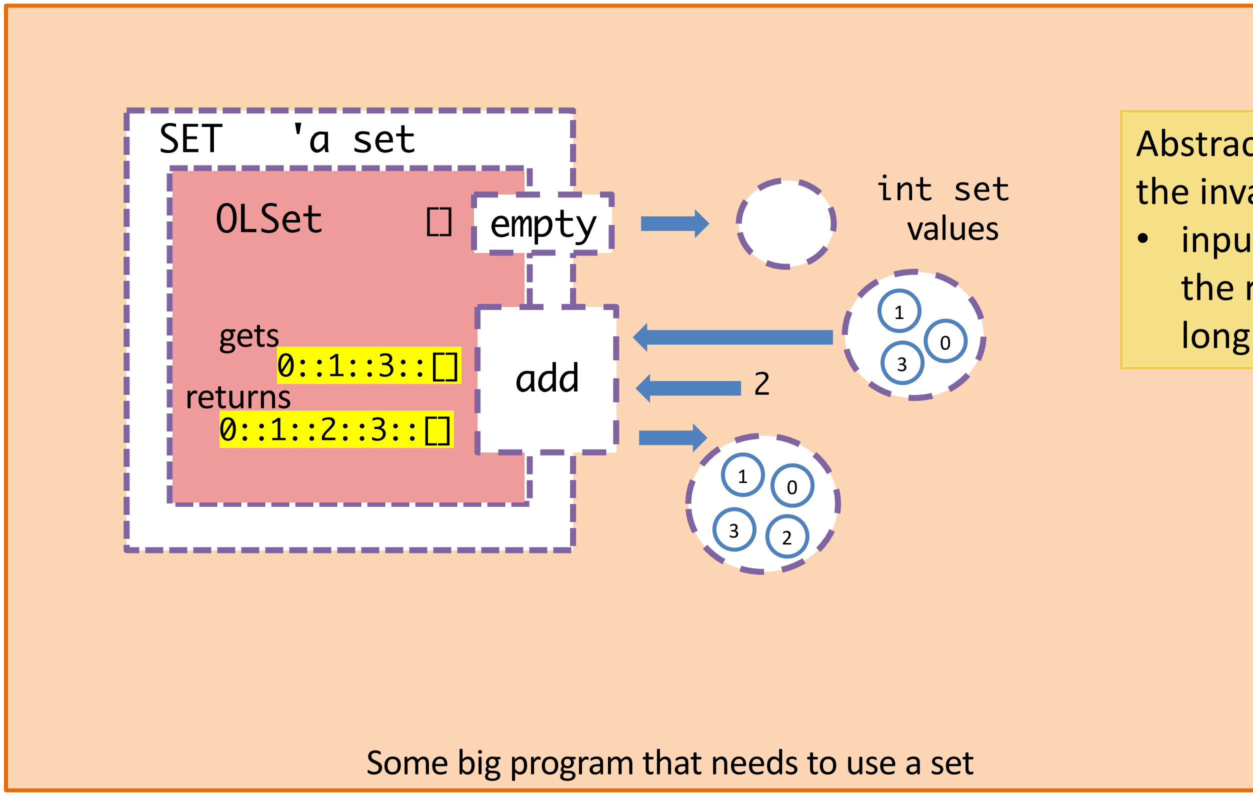
Some big program that needs to use a set

Implementation



Some big program that needs to use a set

Implementation



Abstraction Boundary – "preserves the invariants"

- inputs to the SET module satisfy the representation invariants as long as the created outputs do

11: Given add of type 'a -> 'a set -> 'a set, what does it mean to say that this function "preserves invariants" ?

The output of this function is always a valid set, no matter what inputs are provided. — 0%

If the input set is valid, then the output of this function is always a valid set. — 0%

If the input set is valid, then the output of this function may or may not be a valid set. — 0%

The output of this function is never a valid set, no matter what inputs are provided. — 0%

None of the above — 0%

Given "add" of type 'a -> 'a set -> 'a set, what does it mean to say that this function "preserves invariants" ?

1. The output of this function is always a valid set, no matter what inputs are provided.
2. If the input set is valid, then the output of this function is always a valid set.
3. If the input set is valid, then the output of this function may or may not be a valid set.
4. The output of this function is never a valid set, no matter what inputs are provided.
5. None of the above

Answer: 2

In the module OLSet, does this function "preserve invariants" ?

```
let add (x : 'a) (s : 'a set) : 'a set
   = x :: s
```

1. yes
2. no

Answer: 2

In the module OLSet, does this function "preserve invariants" ?

```
let list_of_set (s : 'a set) : 'a list
   = s
```

1. yes
2. no

Answer: 1

In the module OLSet, does this function "preserve invariants" ?

```
let set_of_list (s : 'a list) : 'a set
   = s
```

1. yes
2. no

Answer: 2

In the module OLSet, does this function "preserve invariants" ?

```
let rec set_of_list (s : 'a list) : 'a set =
  begin match s with
   | [] -> []
   | (h :: t) -> add h (set_of_list t)
  end
```

1. yes
2. no

Answer: 1

What is a good signature?

Fall 2022 Question 3 #324

Anonymous
15 hours ago in **Exams**

PIN STAR WATCH 29 VIEWS

Hi!

Can someone clarify the difference between ill-typed, unusable, invariant and good interfaces?

Thank you!

Comment Edit Delete Endorse •••

1 Answer

Luis Sangueado STAFF
12 hours ago

Sure!

Ill-typed: The type signature for one or more functions has an incorrect type such as trying to add an int and a string together to produce an int

Unusable: The provided type declaration would render an implementation unusable. For example, in HW3, not having the definition of an empty set or the `set_of_list` function would render our interface unusable since we'd never be able to create a set type!

Good Signature: Set

```
module type SET = sig

   type 'a set

   val empty       : 'a set
   val add         : 'a -> 'a set -> 'a set
   val member      : 'a -> 'a set -> bool
   val equals      : 'a set -> 'a set -> bool
   val set_of_list : 'a list -> 'a set

end
```

Unusable Signature: Set

```
module type SET = sig

  type 'a set


  val add        : 'a -> 'a set -> 'a set
  val member     : 'a -> 'a set -> bool
  val equals     : 'a set -> 'a set -> bool


end
```

Type is abstract. All we know about it is what is in the signature. All sets must be constructed from operations listed here.

```
let s = ???
```

Clients have no way of constructing a map
Using [] doesn't type check

Good Signature: Set (Again)

```
module type SET = sig

   type 'a set

   val empty       : 'a set
   val add         : 'a -> 'a set -> 'a set
   val member      : 'a -> 'a set -> bool
   val equals      : 'a set -> 'a set -> bool
   val set_of_list : 'a list -> 'a set

end
```

Unsafe Signature: Set

```
module type SET = sig

   type 'a set = 'a list

   val empty      : 'a set
   val add        : 'a -> 'a set -> 'a set
   val member     : 'a -> 'a set -> bool
   val equals     : 'a set -> 'a set -> bool
   val set_of_list : 'a list -> 'a set

end
```

Invariant: elements are sorted in the list, no duplicates

```
let s = [ "uno" ; "dos" ; "tres" ] in
member s "dos" ?
```

Clients can call module code with sets that don't satisfy the invariant

Good Signature: Set (Again)

```
module type SET = sig

   type 'a set

   val empty       : 'a set
   val add         : 'a -> 'a set -> 'a set
   val member      : 'a -> 'a set -> bool
   val equals      : 'a set -> 'a set -> bool
   val set_of_list : 'a list -> 'a set

end
```

Unsafe Signature: Set

```
module type SET = sig

   type 'a set

   val empty        : 'a set
   val add          : 'a -> 'a set -> 'a set
   val member       : 'a -> 'a list -> bool
   val equals       : 'a set -> 'a set -> bool
   val set_of_list : 'a list -> 'a set

end
```

```
let s = [ "uno" ; "dos" ; "tres" ] in
member s "dos" ?
```

Clients can call module code with sets that don't satisfy the invariant

Good Signature: Set (Again)

```
module type SET = sig

   type 'a set

   val empty       : 'a set
   val add         : 'a -> 'a set -> 'a set
   val member      : 'a -> 'a set -> bool
   val equals      : 'a set -> 'a set -> bool
   val set_of_list : 'a list -> 'a set

end
```

Unimplementable Signature: Set

```
module type SET = sig

   type 'a set = 'a * 'a

   val empty        : set
   val add          : 'a -> 'b set -> 'a set
   val member       : 'a -> 'a set -> bool
   val equals       : 'a set -> 'a set -> bool
   val set_of_list : 'a list -> 'a set

end
```

1. Wrong implementation type --- operations won't satisfy properties
2. Missing type arguments (empty) --- doesn't compile!
3. Type too generic (add)

Files, Signatures and Modules

.ml and .mli files

foo.mli

```
type t
val z : t
val f : t -> int
```

foo.ml

```
type t = int
let z : t = 0
let f (x:t) : int =
  x + 1
```

testFoo.ml

```
;; open Foo
;; print_int
   (Foo.f Foo.z)
```

Files

```
module type FOO = sig
  type t
  val z : t
  val f : t -> int
end

module Foo : FOO = struct
  type t = int
  let z : t = 0
  let f (x:t) : int =
    x + 1
end

module Test = struct
  ;; open Foo
  ;; print_int
      (Foo.f Foo.z)
end
```

The file `foo.ml` defines a module called `Foo`.

The file `foo.mli` defines a signature.

Other modules must use definitions in `Foo` according to its signature.

*If a function **isn't** listed in* `foo.mli`*, then it can only be used in* `foo.ml`*.*

*If a type is **abstract** in* `foo.mli`*, then only* `foo.ml` *knows its concrete definition.*

Property-Based Testing

Testing Styles

- “From the inside”…
  - If we know the concrete representation of our data, we can test the effect of each operation on that representation
  - Necessary for checking that operations preserve invariants

- “From the outside”…
  - If the concrete representation is hidden, this doesn’t work!
  - We need a different way to think about testing

What Should We Test?

- Interface: Names and types of operations on the abstract type
- Properties: How the operations behave and interact
  – “Elements that were added can be found by `lookup`”
  – “Adding an element a second time doesn’t change the elements of a set
  – “Adding elements in a different order doesn’t change the outcome of later operations”

Test the properties!

A *property* is a general statement about the behavior of functions in the interface.

For *any* set s and *any* element x, `member x (add x s) = true`

A good test case *checks a specific instance* of the property:

```
let test () : bool = (member 3 (add 3 empty))
;; run_test "member 3 (add 3 empty)" test
```

Property-based Testing

1. Translate informal requirements into general statements about the interface.

   Example: “Order doesn’t matter” becomes
   For *any* set `s` and *any* elements `x` and `y`,
   `add x (add y s)` "equals" `add y (add x s)`

2. Write tests for the “interesting” instances of the general statement.

   Example “interesting” choices
   - `s` is `empty` *vs.* `s` is nonempty
   - `x = y` *vs.* `x <> y`
   - `x` and/or `y` already in `s` *vs.* `x` and `y` different from what’s in `s`

Notes:

- You usually can't test all possibilities (too many!), so just try to cover the “interesting” choices
- Be careful with equality! `ULSet.equals` and BSTSet.equals are *not* the same as "="

Finite Maps

A case study on abstract interfaces

and concrete implementations

Motivating Scenario

- Suppose you were writing a course-management system and needed to look up the lab section for a student given the student's PennKey…
  - Students might add/drop the course
  - Students might switch lab sections
  - Students should be in only *one* lab section

- How would you do it? What data structure would you use?

Key/Value store

Key	Value
“stephanie”	15
“mitch”	05
“ezaan”	10
“likat”	15
…	…

- Each key is associated with a value.
  - No two keys are identical
  - Values can be repeated
- Given the key “stephanie”, we want to find / lookup the value 15

Finite Maps

Design Process Step 1: Understand the problem

- A *finite map* (a.k.a. *dictionary*) is a collection of *entries* from distinct *keys* to *values*.
  - Operations to *add* a new entry, *test* for key membership, *get* the value bound to a particular key, *list* all entries stored in the map
- Example: we might use a finite map to look up the lab section of a CIS 1200 student
- Like sets, *finite maps* appear in many settings:
  - domain names to IP addresses
  - words to their definitions (a dictionary)
  - user names to passwords
  - …

Signature: Finite Map

Design Process Step 2: specify the interface

```
module type MAP = sig

  type ('k,'v) map

  val empty    : ('k,'v) map
  val add      : 'k -> 'v -> ('k,'v) map -> ('k,'v) map
  val remove   : 'k -> ('k,'v) map -> ('k,'v) map
  val mem      : 'k -> ('k,'v) map -> bool
  val get      : 'k -> ('k,'v) map -> 'v
  val equals   : ('k,'v) map -> ('k,'v) map -> bool

end
```

The *map* type is generic in *two* ways: type of keys and type of values

Properties of Finite Maps

Design Process Step 3: write test cases

For any finite map m, key k, and value v:

1. `get k (add k v m) = v`
2. If `k1 <> k2` then
   `get k1 (add k2 v2 (add k1 v1 m)) = v1`
3. If `mem k m = true` then
   there is a `v` such that `get k m = v`
4. If `mem k m = false` then
   `get k m = v` fails
5. `mem k (add k v m) = true`

(among others…)

Tests for Finite Map abstract type

Design Process Step 3: write test cases

```
;; open Assert

(* Specifying the properties of the MAP abstract type via test cases. *)

(* A simple map with one element. *)
let m1 : (int,string) map = add 1 "uno" empty

(* access value for key in the map *)
;; run_test "find 1 m1" (fun () -> (get 1 m1) = "uno")

(* find for value that does not exist in the map? *)
;; run_failing_test "find 2 m1" (fun () -> (get 2 m1) = "dos" )

let m2 : (int, string) map = add 1 "un" m1

(* find after redefining value, should be new value *)
;; run_test "find 1 m2" (fun () -> (get 1 m2) = "un")

(* test membership *)
;; run_test "mem test" (fun () ->
        mem 1 (add 2 "dos" (add 1 "uno" empty)))
```

Using an anonymous function avoids making up a (redundant) function name for the test

Finite Map Demo

Implementing the module

finiteMap.ml

Implementation: Ordered Lists

Design Process Step 4: implement it!

```
module Assoc : MAP = struct
  (* Represent a finite map as a list of pairs.      *)
  (* Representation invariant:                       *)
  (*   - no duplicate keys (helps get, remove)       *)
  (*   - keys are sorted (helps equals, get)         *)

  type ('k,'v) map = ('k * 'v) list

  let empty : ('k,'v) map = []

  let rec mem (key:'k) (m : ('k,'v) map) : bool =
    begin match m with
    | [] -> false
    | (k,v)::rest ->
      (key >= k) &&
         ((key = k) || (mem key rest))
    end
```

Implementation: Ordered Lists

```
let rec get (key:'k) (m : ('k,'v) map) : 'v =
  begin match m with
  | [] -> failwith "key not found"
  | (k,v)::rest ->
    if key < k then failwith "key not found"
    else if key = k then v
    else get key rest
  end

let rec remove (key:'k) (m : ('k,'v) map) : ('k,'v) map =
  begin match m with
  | [] -> []
  | (k,v)::rest ->
    if key < k then m
    else if key = k then rest
    else (k,v)::remove key rest
  end
```

Summary: Abstract Types

- Different programming languages support different ways of defining abstract types

- At a minimum, this means providing:
  - A way to specify (write down) an interface
  - A means of hiding implementation details (*encapsulation*)

- In OCaml:
  - Interfaces are specified using a *signature* or *interface*
  - Encapsulation: the interface can *omit* information
    - type definitions
    - names of auxiliary functions
  - Clients *cannot* mention values or types not named in the interface

Typechecking

How does OCaml* typecheck your code?

*Historical aside: the algorithm we are about to see is known as the Damas-Hindley-Milner type inference algorithm. Turing Award winner Robin Milner was, among other things, the inventor of "ML" (for "meta language"), from which OCaml gets its "ml".

OCaml Typechecking Errors

```
type ('k,'v) map = ('k * 'v) list

(* A finite map that contains no entries. *)
let empty () = []

let rec me
  begin ma
  | [] ->
  | (k,v):
    if key
    (key = k) || (mem key rest)
  end

;; run_test "mem test" (fun () ->
  mem "b" [("a",3); ("b",4)]
)

let rec get (key:'k) (m : ('k,'v) map) : 'v =
  begin match m with
  | [] -> failwith "not found"
```

```
Signature mismatch:
       ...
       Values do not match:
         val empty : unit -> 'a list
       is not included in
         val empty : ('k, 'v) map
       File "finiteMap.ml", line 13, characters 2-27: Expected
declaration
       File "finiteMap.ml", line 60, characters 6-11: Actual declaration
```

Typechecking

How do we determine the type of an expression?

1. Recursively determine the types of *all* sub-expressions
   - Constants have "obvious" types

     `3 : int`   `"foo" : string`   `true : bool`
   - Identifiers may have type annotations
     - `let` and function arguments
     - Module signatures/interfaces

2. Expressions that *construct* structured values have compound types built from the types of sub-expressions

```
(3, "foo")                      : int * string
(fun (x:int) -> x + 1)          : int -> int
Node(Empty, (3, "foo"), Empty) : (int * string) tree
```

Typechecking Functions

To typecheck a function:

```
fun (x:int) -> x + x
```

??

Typechecking Functions

To typecheck a function:

```
fun (x:int) -> x + x
```

T_{arg} -> T_{ans}

Make up "new names" for the input (argument) and output (answer) types.

Typechecking Functions

To typecheck a function:

```
fun (x:int) -> x + x
```

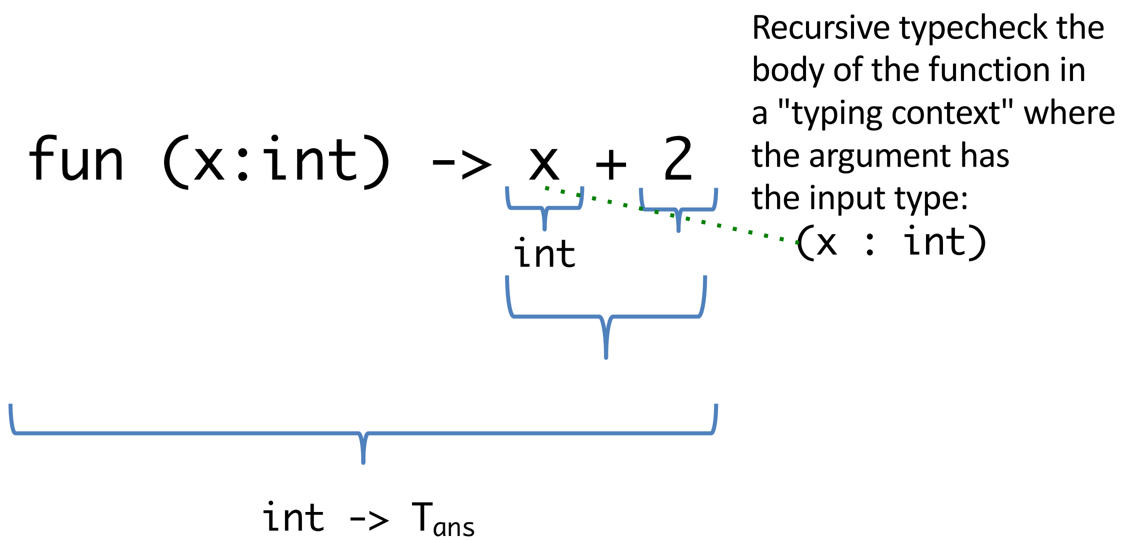
Take the argument type from the type annotation (if any*): T_{arg} = `int`

T_{arg} -> T_{ans}

*If there is no annotation, just use the "fresh" name…

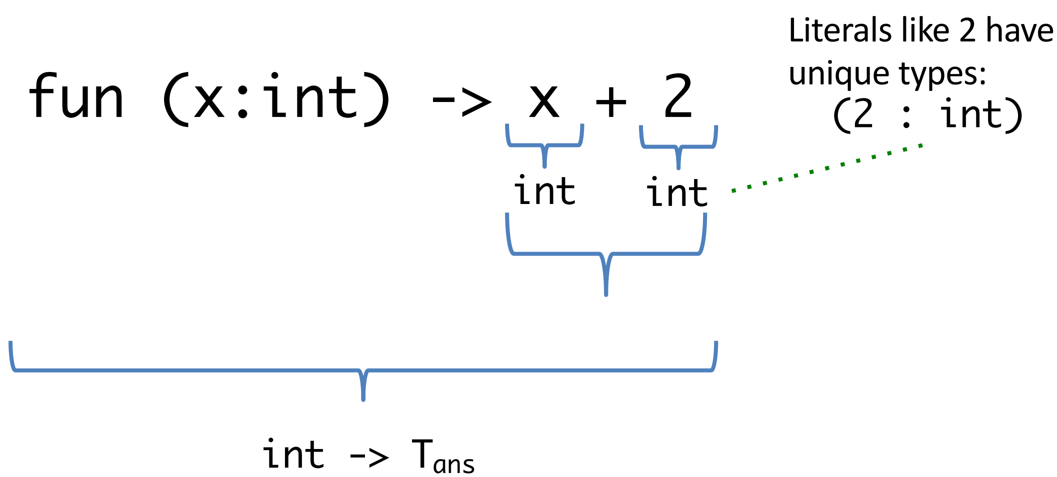
Typechecking Functions

To typecheck a function:



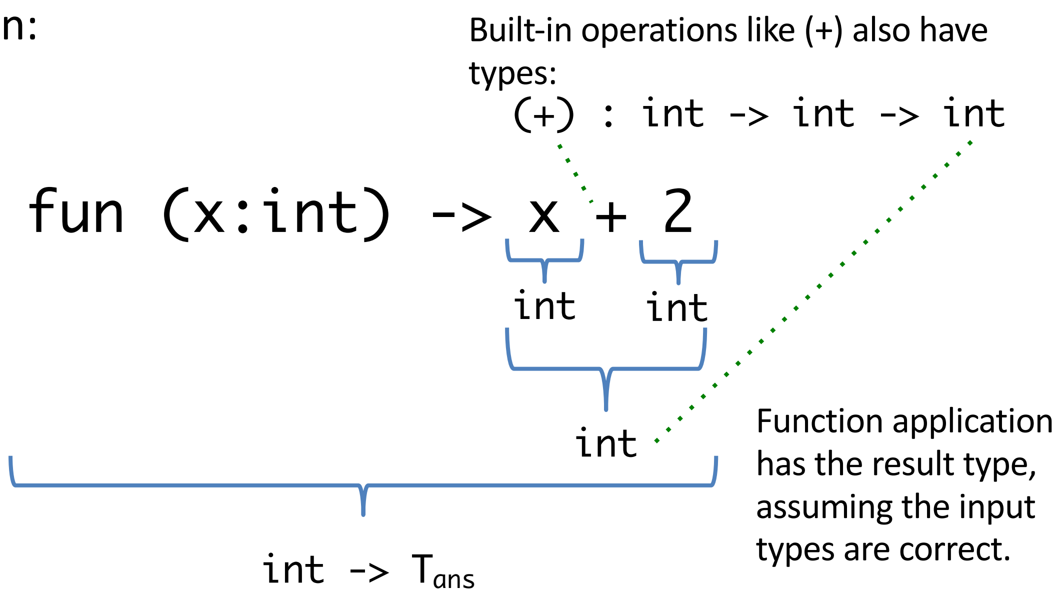
Typechecking Functions

To typecheck a function:



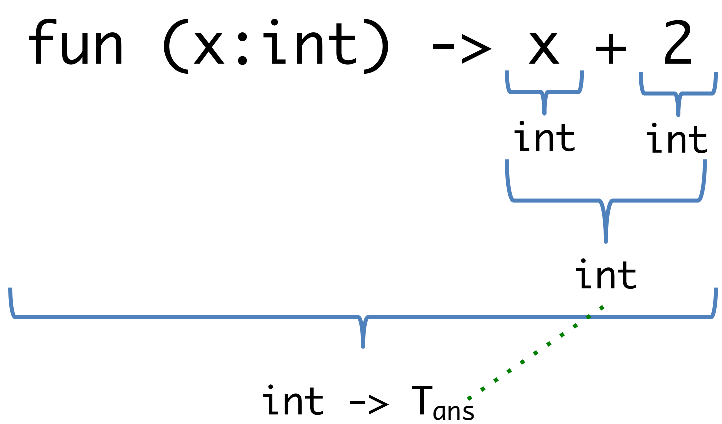
Typechecking Functions

To typecheck a function:



Typechecking Functions

To typecheck a function:

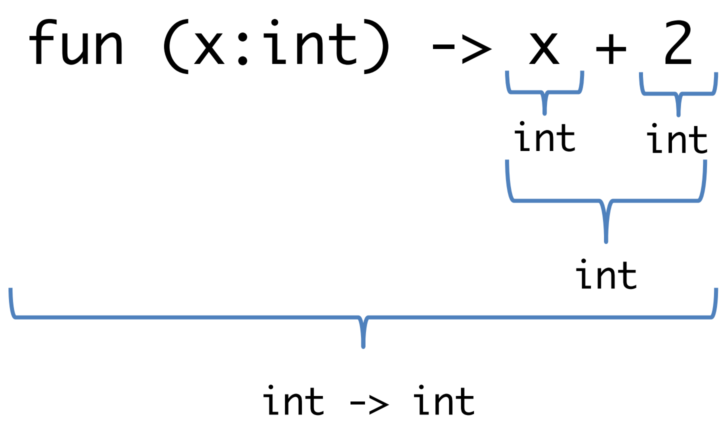


The "answer" type is the type of the body.

T_{ans} = int

Typechecking Functions

To typecheck a function:



Typechecking II

3. The type of a function-application expression is obtained as the result from the function type:
   - Given a function `f` : T_{arg} -> T_{ans}
   - and an argument `e` : T_{arg} of the input type
   - the application `(f e)` : T_{ans} has the answer type

```
((fun (x:int) (y:bool) -> y)  3)  : ??
```

Typechecking II

3. The type of a function-application expression is obtained as the result from the function type:
   - Given a function `f` : T_{arg} `->` T_{ans}
   - and an argument `e` : T_{arg} of the input type
   - the application `(f e)` : T_{ans} has the answer type

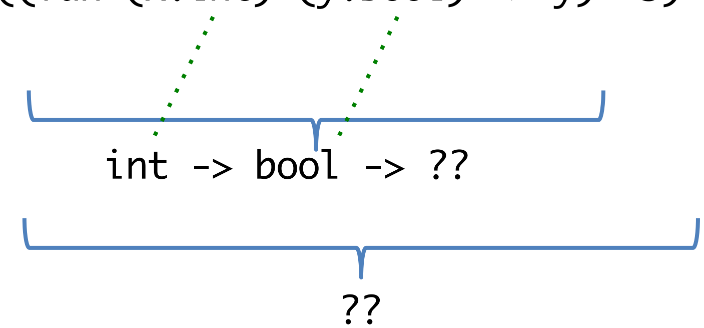
```
((fun (x:int) (y:bool) -> y)  3)  : ??
```

??

Typechecking II

3. The type of a function-application expression is obtained as the result from the function type:
   - Given a function `f` : T_{arg} `->` T_{ans}
   - and an argument `e` : T_{arg} of the input type
   - the application `(f e)` : T_{ans} has the answer type

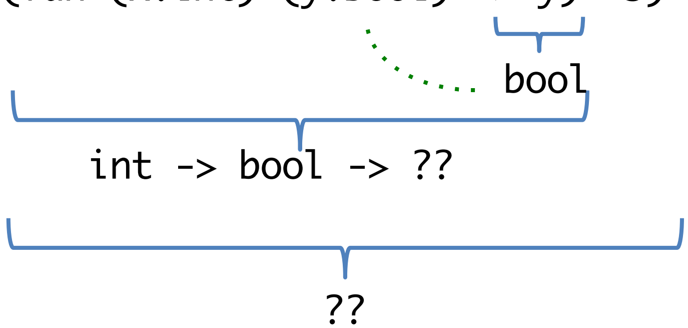
```
((fun (x:int) (y:bool) -> y)  3)  : ??
```



Typechecking II

3. The type of a function-application expression is obtained as the result from the function type:
   - Given a function `f` : T_{arg} -> T_{ans}
   - and an argument `e` : T_{arg} of the input type
   - the application `(f e)` : T_{ans} has the answer type

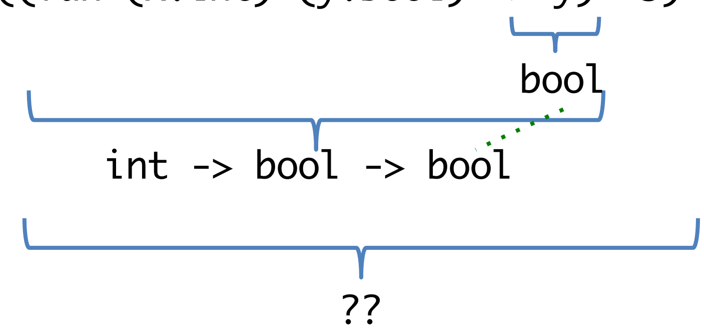
```
((fun (x:int) (y:bool) -> y)  3)  : ??
```



Typechecking II

3. The type of a function-application expression is obtained as the result from the function type:
   - Given a function `f` : T_{arg} `->` T_{ans}
   - and an argument `e` : T_{arg} of the input type
   - the application `(f e)` : T_{ans} has the answer type

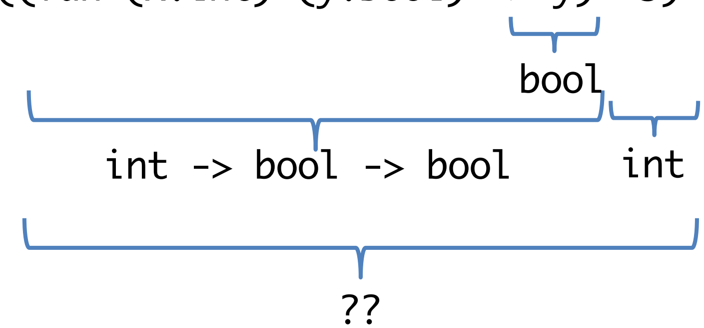
```
((fun (x:int) (y:bool) -> y)  3)  : ??
```



Typechecking II

3. The type of a function-application expression is obtained as the result from the function type:
   - Given a function `f` : T_{arg} `->` T_{ans}
   - and an argument `e` : T_{arg} of the input type
   - the application `(f e)` : T_{ans} has the answer type

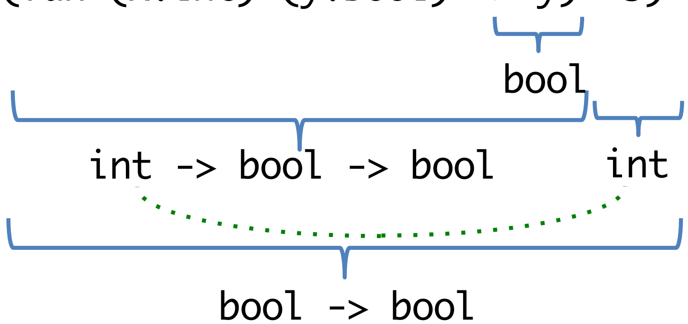
```
((fun (x:int) (y:bool) -> y)  3)  : ??
```



Typechecking II

3. The type of a function-application expression is obtained as the result from the function type:
   - Given a function `f` : T_{arg} `->` T_{ans}
   - and an argument `e` : T_{arg} of the input type
   - the application `(f e)` : T_{ans} has the answer type

```
((fun (x:int) (y:bool) -> y)  3)  : ??
```



Here:
T_1 = `int`
T_2 = `bool -> bool`

Typechecking III

- What about generics? i.e., what if `f:'a ->'a`?
- For generic types we *unify*
  - Given a function `f` : T_1 -> T_2
  - and an argument `e` : U_1 of the input type

  Can *"match up"* T_1 and U_1 to obtain information about type parameters in T_1 and U_1 based on their usage
- *Unification:*
  - try to match up corresponding parts of the type

    `(int list) tree ⇔ 'a tree`
  - Obtain an *instantiation*: e.g. `'a = int list`
  - *Propagate* that information to all occurrences of `'a`
  - If not possible, unification fails, meaning a type checking error

    `bool tree ⇔ int tree`

    ERROR! `bool ≠ int`

Example Typechecking Problem

```
empty   : ('k, 'v) map
add     : 'k -> 'v -> ('k, 'v) map -> ('k, 'v) map
entries : ('k, 'v) map -> ('k * 'v) list
```

```
fun (x:'v) -> entries (add 3 x empty)
```

??

Example Typechecking Problem

```
empty   : ('k, 'v) map
add     : 'k -> 'v -> ('k, 'v) map -> ('k, 'v) map
entries : ('k, 'v) map -> ('k * 'v) list
```

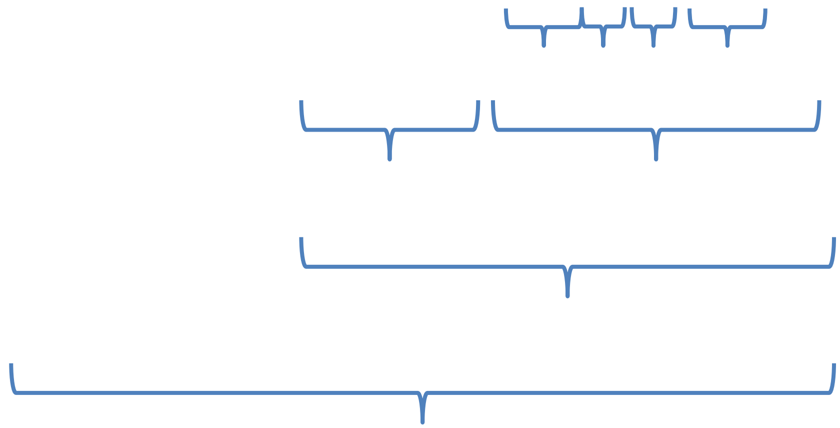
```
fun (x:'v) -> entries (add 3 x empty)
```

```
'v -> ??
```

Example Typechecking Problem

```
empty   : ('k, 'v) map
add     : 'k -> 'v -> ('k, 'v) map -> ('k, 'v) map
entries : ('k, 'v) map -> ('k * 'v) list
```

```
fun (x:'v) -> entries (add 3 x empty)
```

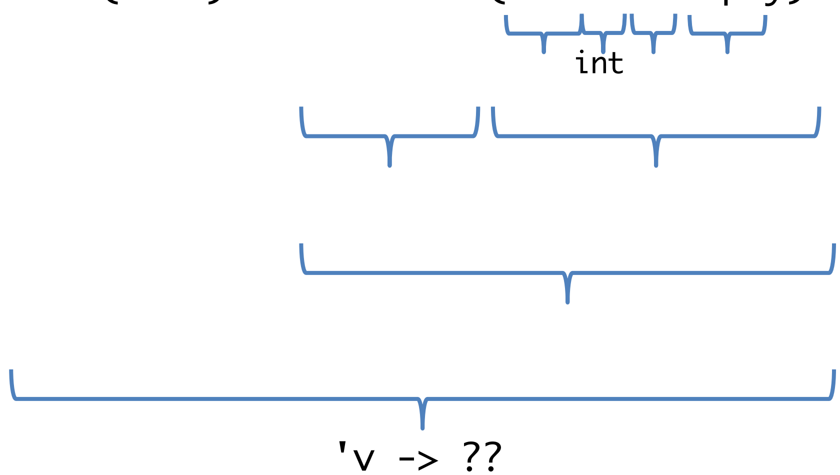


```
'v -> ??
```

Example Typechecking Problem

```
empty   : ('k, 'v) map
add     : 'k -> 'v -> ('k, 'v) map -> ('k, 'v) map
entries : ('k, 'v) map -> ('k * 'v) list
```

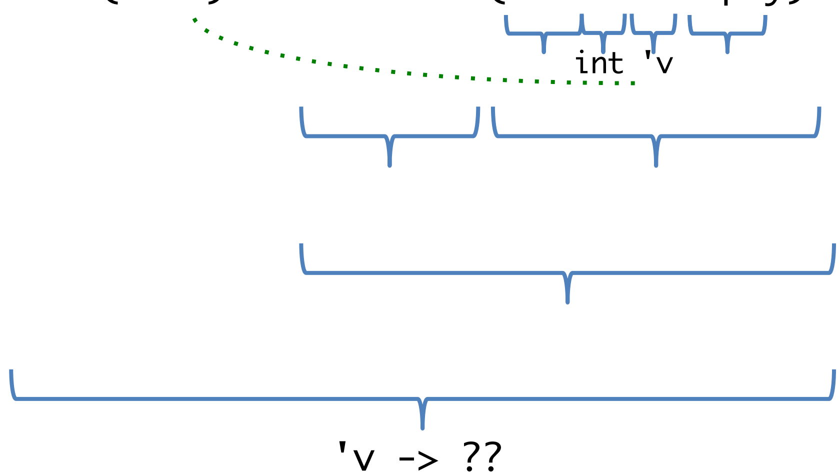
```
fun (x:'v) -> entries (add 3 x empty)
```



Example Typechecking Problem

```
empty   : ('k, 'v) map
add     : 'k -> 'v -> ('k, 'v) map -> ('k, 'v) map
entries : ('k, 'v) map -> ('k * 'v) list
```

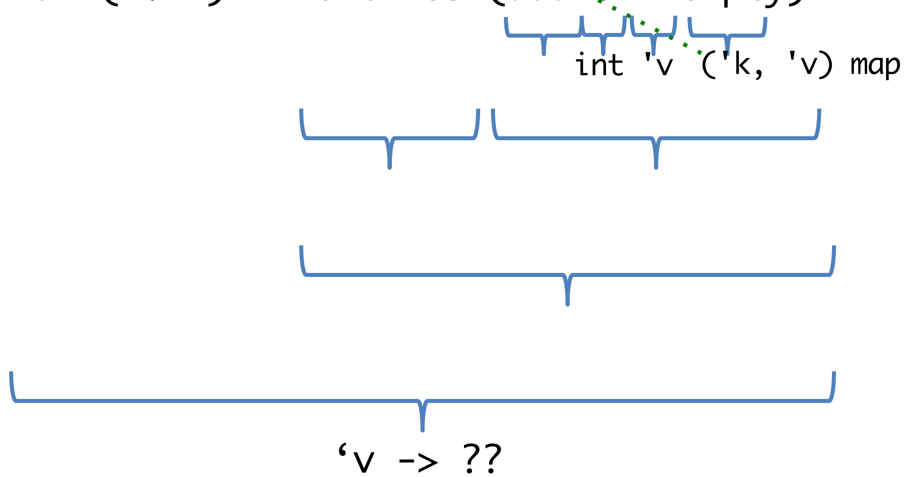
```
fun (x:'v) -> entries (add 3 x empty)
```



Example Typechecking Problem

```
empty   : ('k, 'v) map
add     : 'k -> 'v -> ('k, 'v) map -> ('k, 'v) map
entries : ('k, 'v) map -> ('k * 'v) list
```

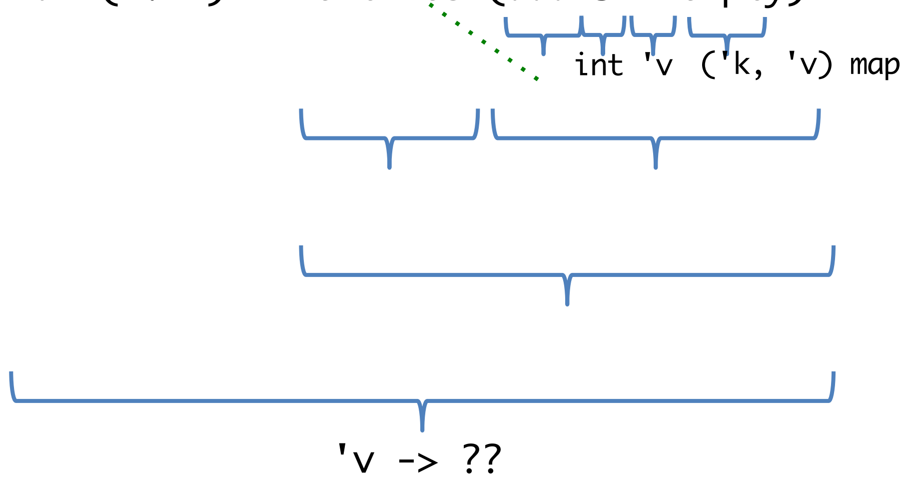
```
fun (x:'v) -> entries (add 3 x empty)
```



Example Typechecking Problem

```
empty   : ('k, 'v) map
add     : 'k -> 'v -> ('k, 'v) map -> ('k, 'v) map
entries : ('k, 'v) map -> ('k * 'v) list
```

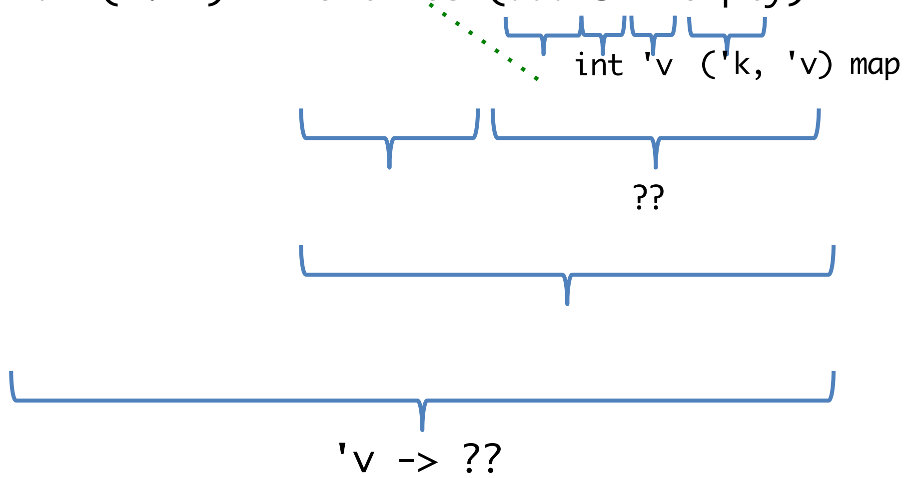
```
fun (x:'v) -> entries (add 3 x empty)
```



Example Typechecking Problem

```
empty   : ('k, 'v) map
add     : 'k -> 'v -> ('k, 'v) map -> ('k, 'v) map
entries : ('k, 'v) map -> ('k * 'v) list
```

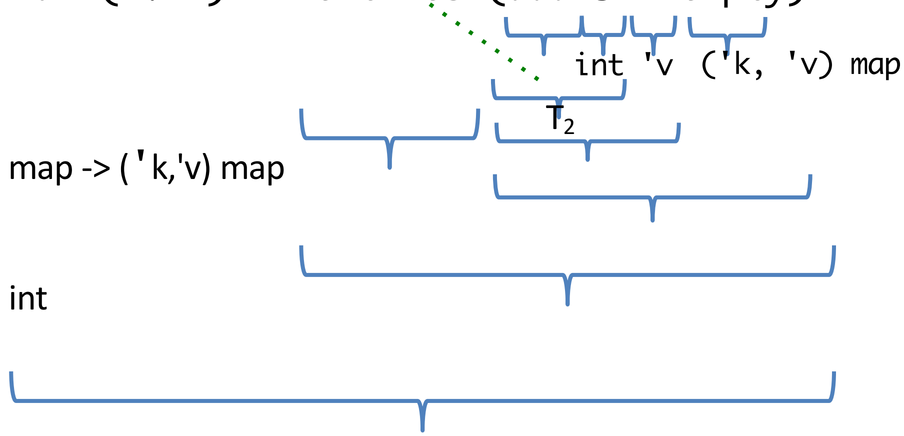
```
fun (x:'v) -> entries (add 3 x empty)
```



Example Typechecking Problem

```
empty   : ('k, 'v) map
add     : 'k -> 'v -> ('k, 'v) map -> ('k, 'v) map
entries : ('k, 'v) map -> ('k * 'v) list
```

```
fun (x:'v) -> entries (add 3 x empty)
```



Application:
T_1 = 'k
T_2 = 'v -> ('k,'v) map -> ('k,'v) map

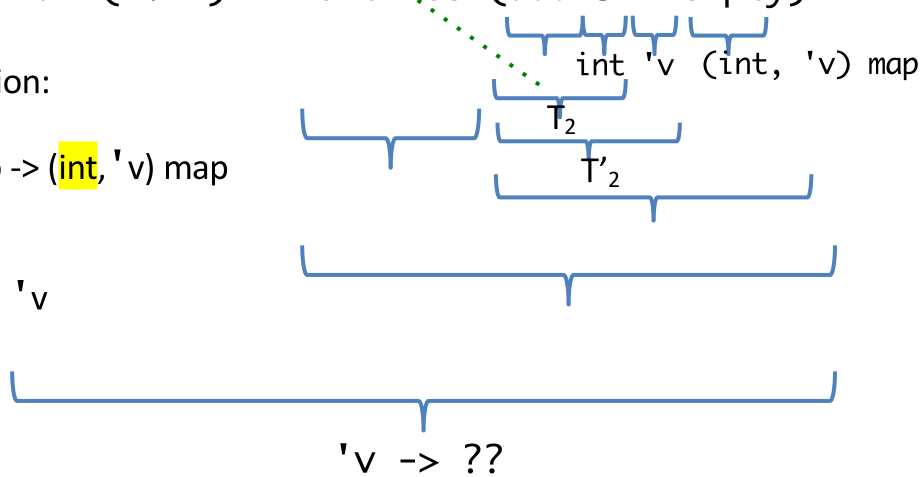
Instantiate: 'k = int

'v -> ??

Example Typechecking Problem

```
empty   : ('k, 'v) map
add     : 'k -> 'v -> ('k, 'v) map -> ('k, 'v) map
entries : ('k, 'v) map -> ('k * 'v) list
```

```
fun (x:'v) -> entries (add 3 x empty)
```



Another Application:

T'₁ = 'v

T'₂ = (int, 'v) map -> (int, 'v) map

Instantiate: 'v = 'v

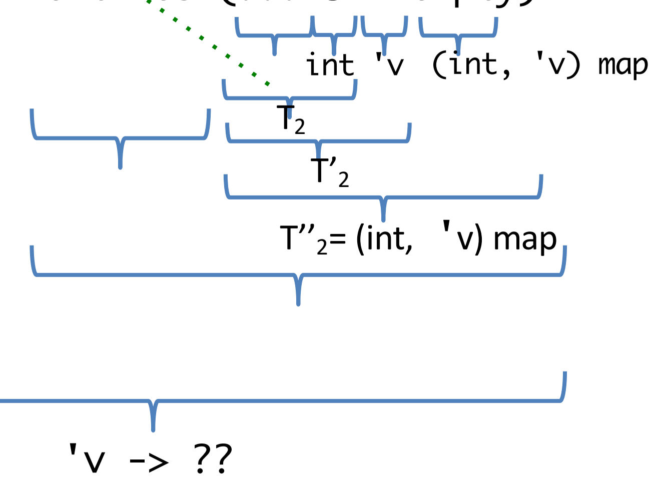
Example Typechecking Problem

```
empty   : ('k, 'v) map
add     : 'k -> 'v -> ('k, 'v) map -> ('k, 'v) map
entries : ('k, 'v) map -> ('k * 'v) list
```

```
fun (x:'v) -> entries (add 3 x empty)
```

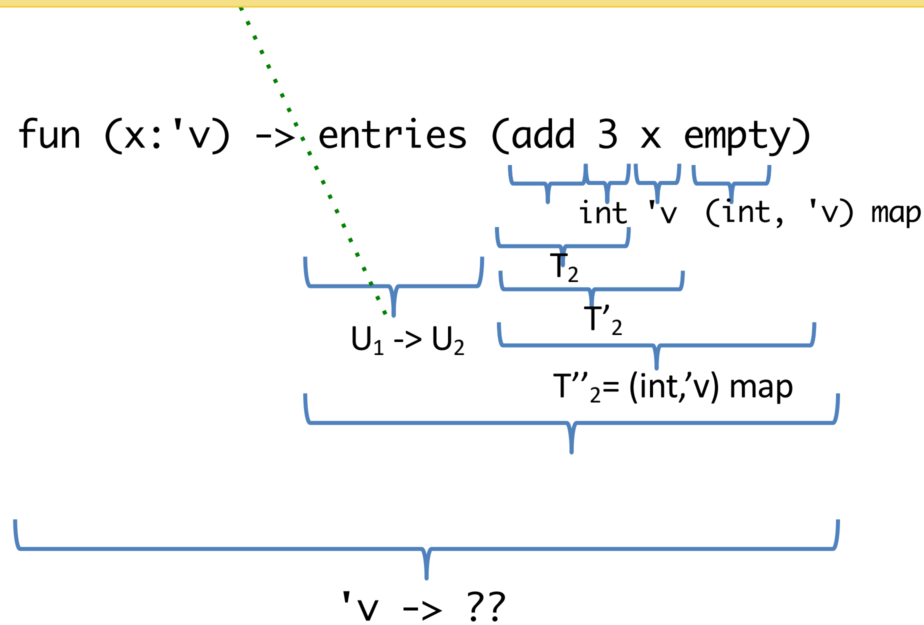
A third Application:
T''$_1$ = (int, 'v) map
T''$_2$ = (int, 'v) map

Argument and argument type already agree



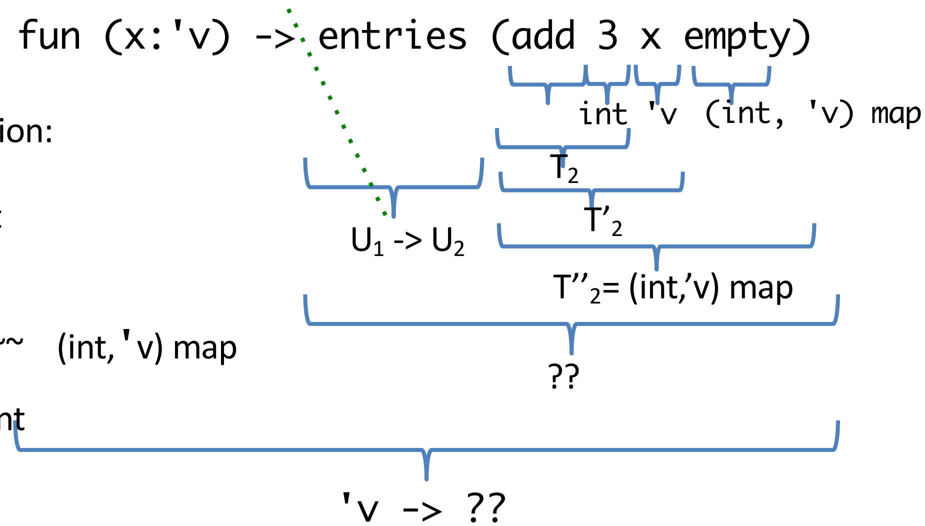
Example Typechecking Problem

```
empty   : ('k, 'v) map
add     : 'k -> 'v -> ('k, 'v) map -> ('k, 'v) map
entries : ('k, 'v) map -> ('k * 'v) list
```



Example Typechecking Problem

```
empty   : ('k, 'v) map
add     : 'k -> 'v -> ('k, 'v) map -> ('k, 'v) map
entries : ('k, 'v) map -> ('k * 'v) list
```



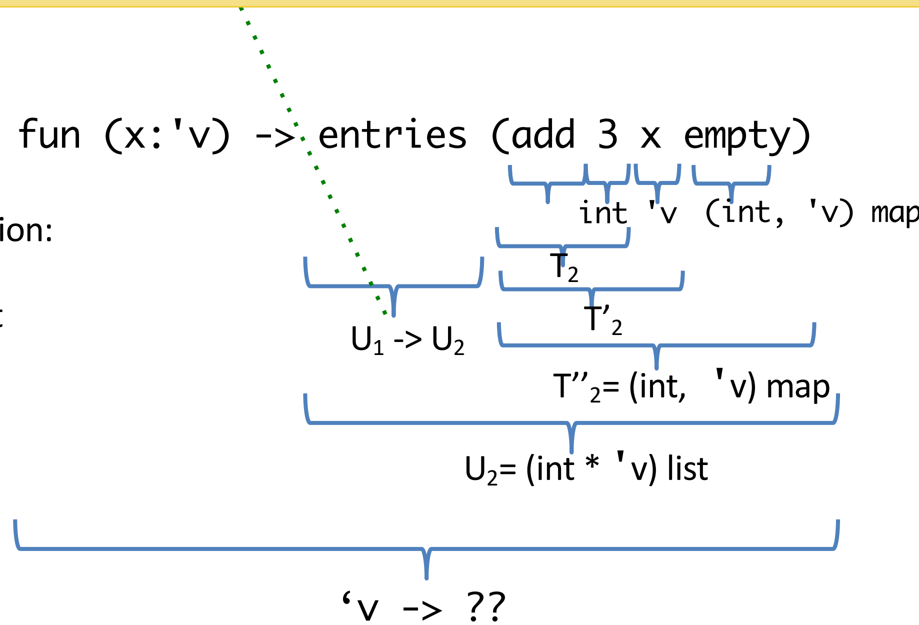
Another Application:
U_1 = ('k,'v) map
U_2 = ('k * 'v) list

Unify U_1 with T''_2
('k,'v) map ~~ (int,'v) map

Instantiate 'k = int

Example Typechecking Problem

```
empty   : ('k, 'v) map
add     : 'k -> 'v -> ('k, 'v) map -> ('k, 'v) map
entries : ('k, 'v) map -> ('k * 'v) list
```

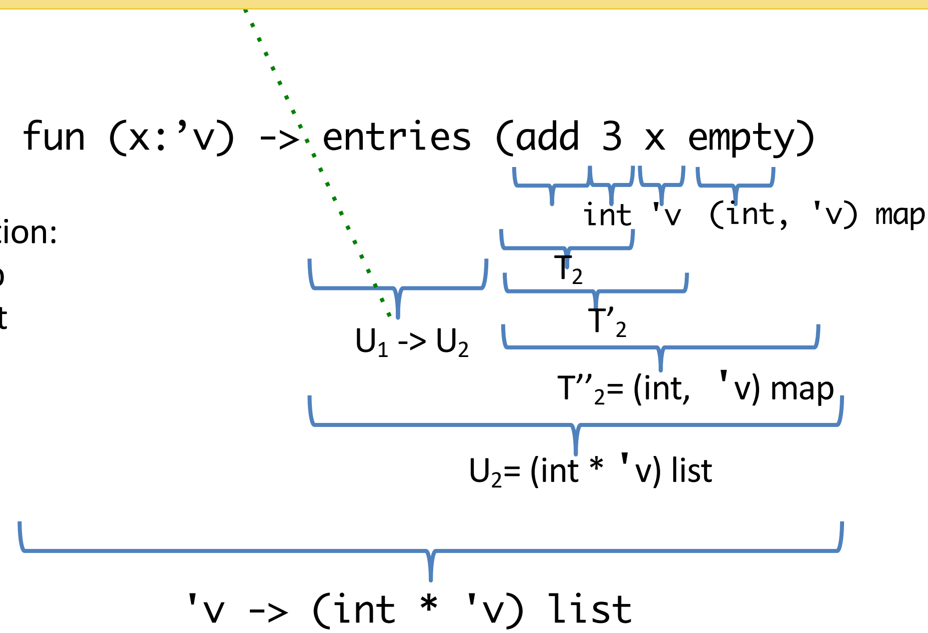


Another Application:
U_1 = (int, 'v) map
U_2 = (int * 'v) list

Example Typechecking Problem

```
empty   : ('k, 'v) map
add     : 'k -> 'v -> ('k, 'v) map -> ('k, 'v) map
entries : ('k, 'v) map -> ('k * 'v) list
```

Another Application:
U_1 = (int, 'v) map
U_2 = (int * 'v) list



Ill-typed Expressions?

- An expression is ill-typed if, during this type checking process, inconsistent constraints are encountered:

```
empty   : ('k, 'v) map
add     : 'k -> 'v -> ('k, 'v) map -> ('k, 'v) map
entries : ('k, 'v) map -> ('k * 'v) list
```

```
add 3 true (add “foo” false empty)
```

Error: found `int` but expected `string`

12: What is the type of this expression?

0

```
let e : ______ =
  transform (fun x y -> x + y)
```

int list -> int list

0%

int list -> int list -> int list

0%

int list -> (int -> int) list

0%

None (it doesn't typecheck)

0%

What is the type of this expression?

```
let e : ______  =
  transform (fun x y -> x + y)
```

1. int list -> int list
2. int list -> int list -> int list
3. int list -> (int -> int) list
4. None (it doesn't typecheck)

Answer: 3