Programming Languages and Techniques (CIS120)

Lecture 7

Binary Search Trees

(Chapters 7 & 8)

Recap: Ordered Trees

Big idea: find things faster by searching less

Key Insight:

Ordered data can be searched more quickly

- This is why telephone books are arranged alphabetically
- Requires the ability to focus on (roughly) *half* of the current data

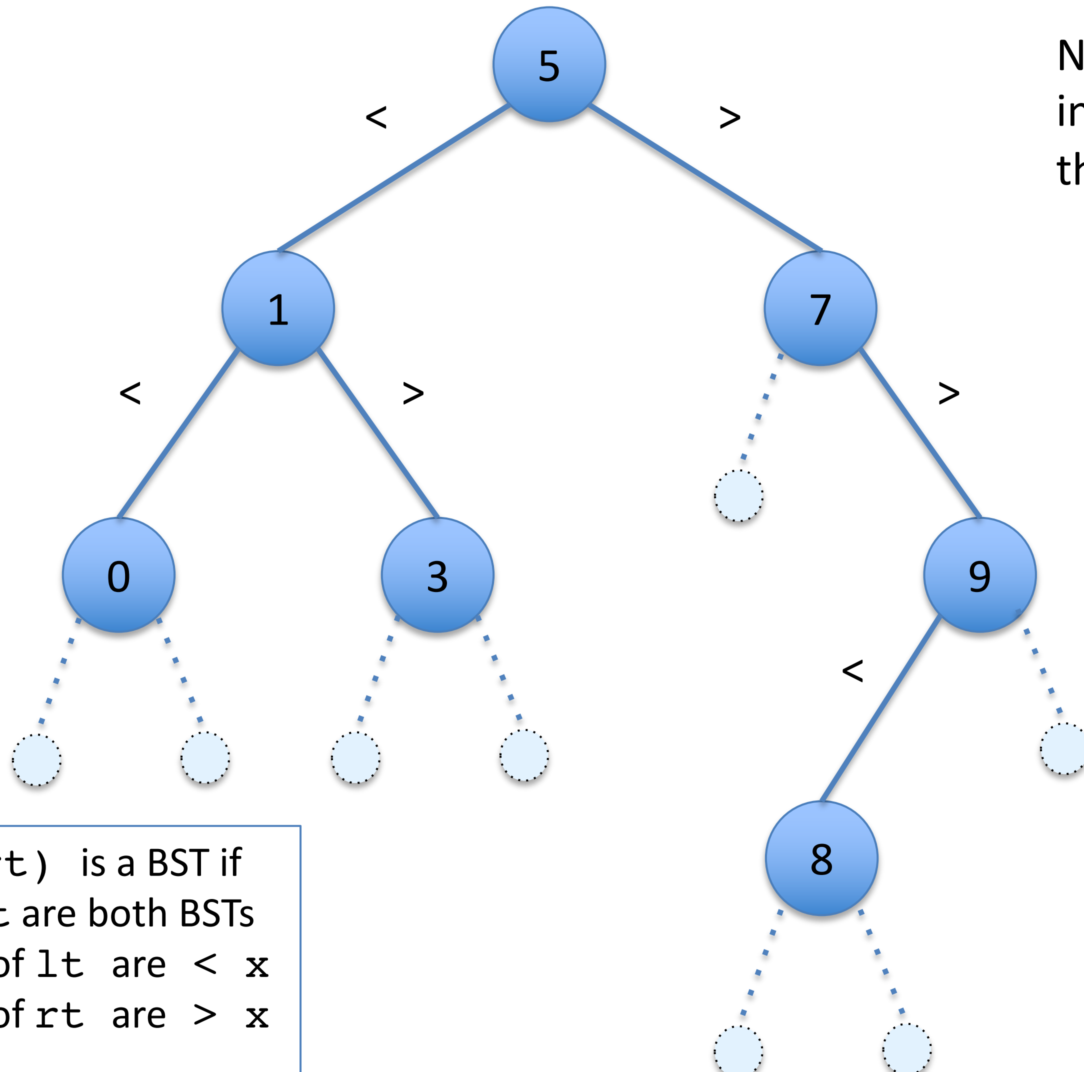
Binary Search Trees

- A *binary search tree* (BST) is a binary tree with some additional *invariants:*

> - `Node(lt,x,rt)` is a BST if
>   - `lt` and `rt` are both BSTs
>   - all nodes of `lt` are $<$ `x`
>   - all nodes of `rt` are $>$ `x`
> - `Empty` is a BST

- *The BST invariant means that container functions can take time proportional to the **height** instead of the **size** of the tree.*

An Example Binary Search Tree

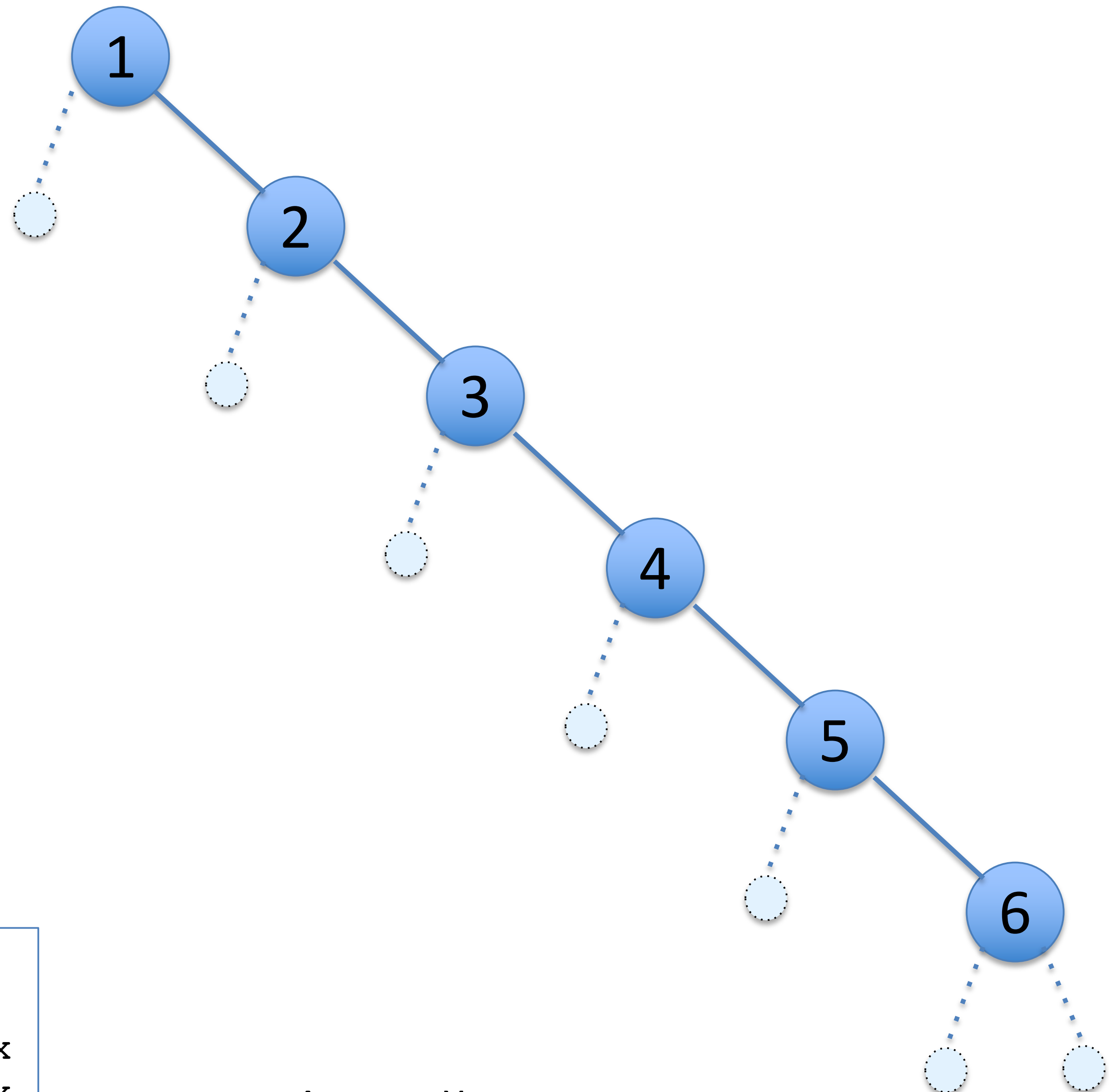


Note that the BST invariants hold for this tree.

- `Node(lt,x,rt)` is a BST if
  - `lt` and `rt` are both BSTs
  - all nodes of `lt` are `<` `x`
  - all nodes of `rt` are `>` `x`
- `Empty` is a BST

Is this a BST??

1. yes
2. no

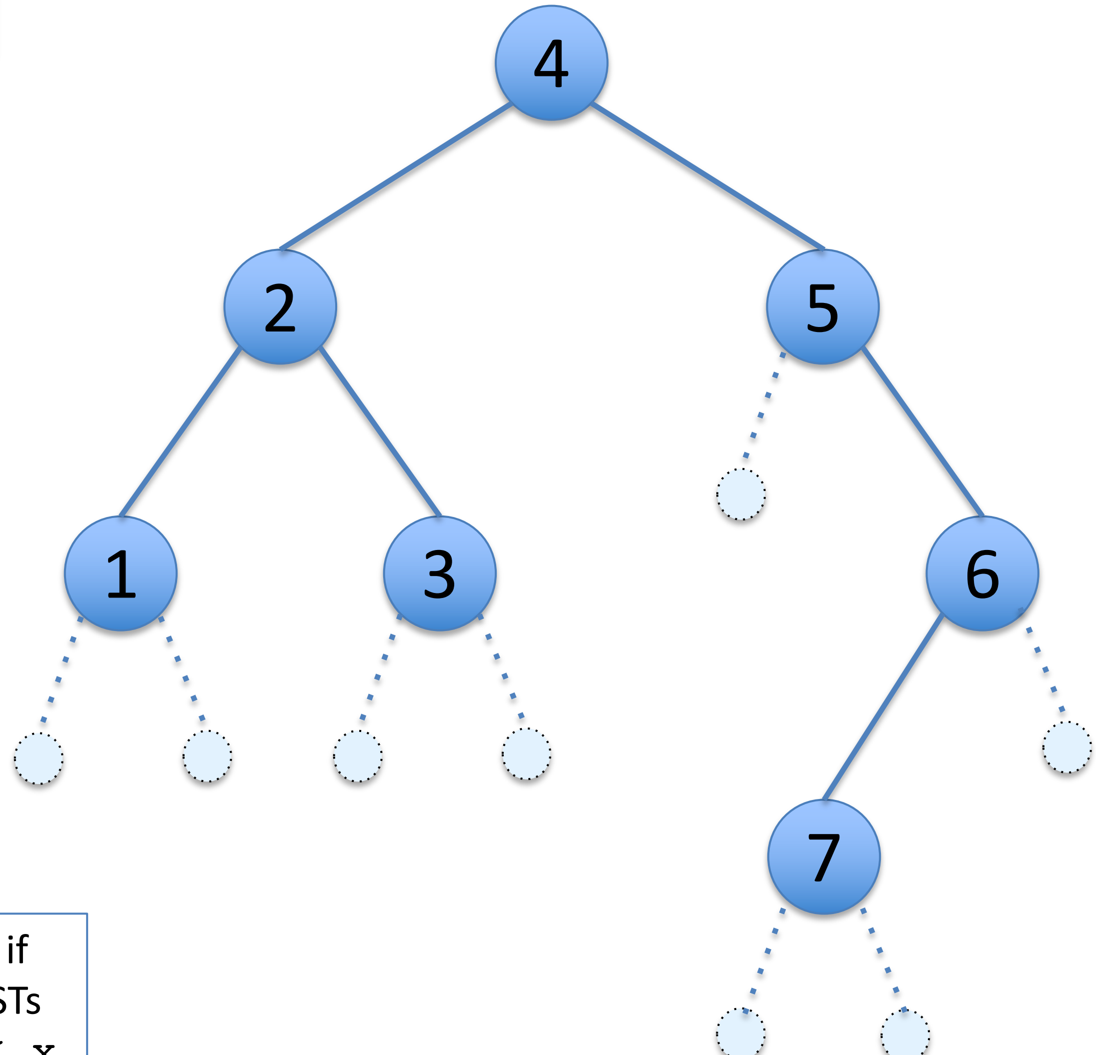


- `Node(lt,x,rt)` is a BST if
  - `lt` and `rt` are both BSTs
  - all nodes of `lt` are < `x`
  - all nodes of `rt` are > `x`
- `Empty` is a BST

Answer: Yes

Is this a BST??

1. yes
2. no

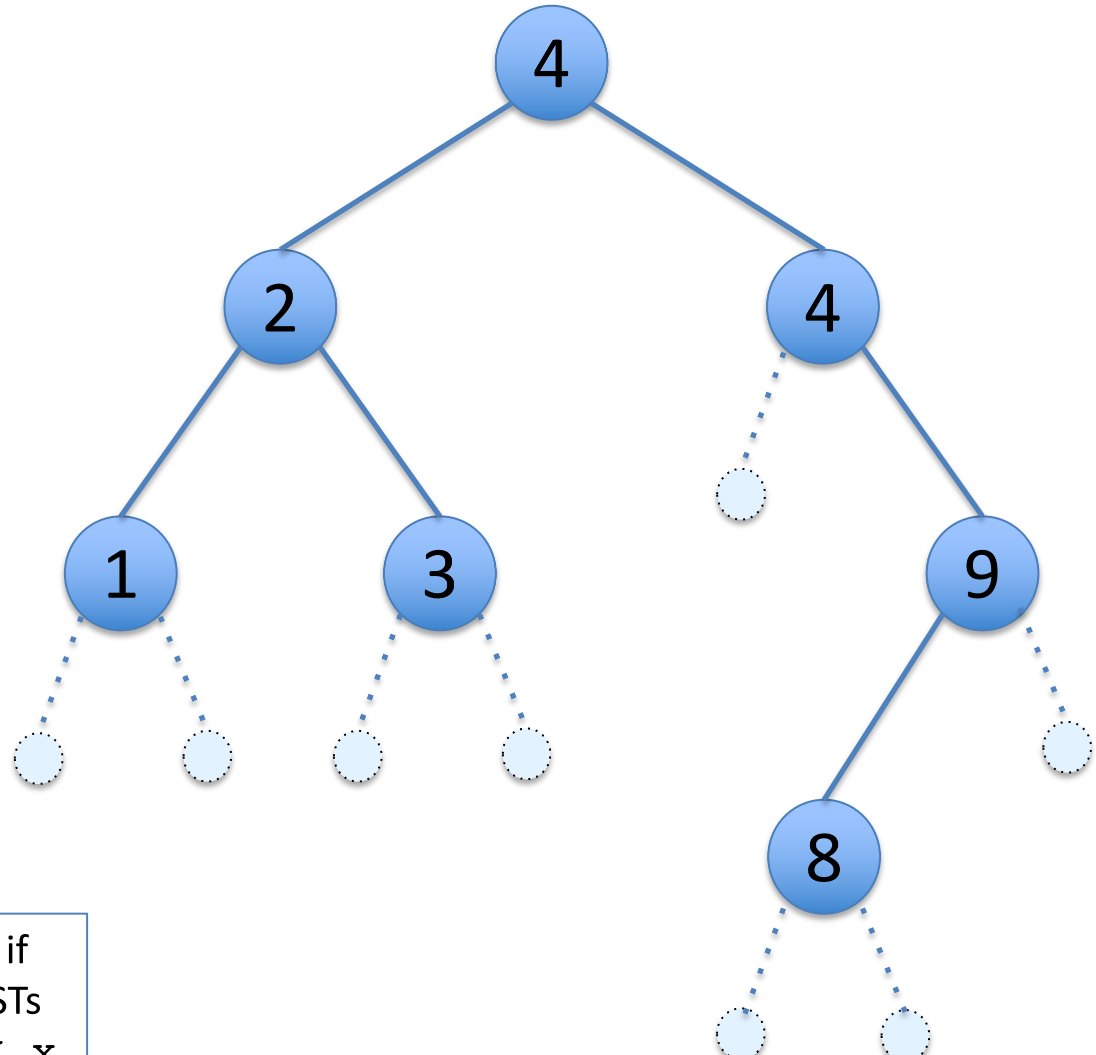


- `Node(lt,x,rt)` is a BST if
  - `lt` and `rt` are both BSTs
  - all nodes of `lt` are `<` `x`
  - all nodes of `rt` are `>` `x`
- `Empty` is a BST

Answer: no, 7 to the left of 6

Is this a BST??

1. yes
2. no

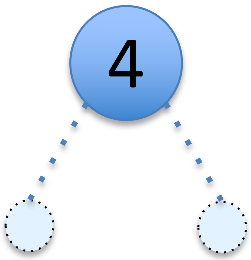


- `Node(lt,x,rt)` is a BST if
  - `lt` and `rt` are both BSTs
  - all nodes of `lt` are `<` `x`
  - all nodes of `rt` are `>` `x`
- `Empty` is a BST

Answer: no, 4 to the right of 4

Is this a BST??

1. yes
2. no



- `Node(lt,x,rt)` is a BST if
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  - all nodes of `lt` are `<` `x`
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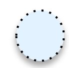
- `Empty` is a BST

Answer: yes

Is this a BST??

1. yes
2. no



- `Node(lt,x,rt)` is a BST if
  - `lt` and `rt` are both BSTs
  - all nodes of `lt` are `<` `x`
  - all nodes of `rt` are `>` `x`
- `Empty` is a BST

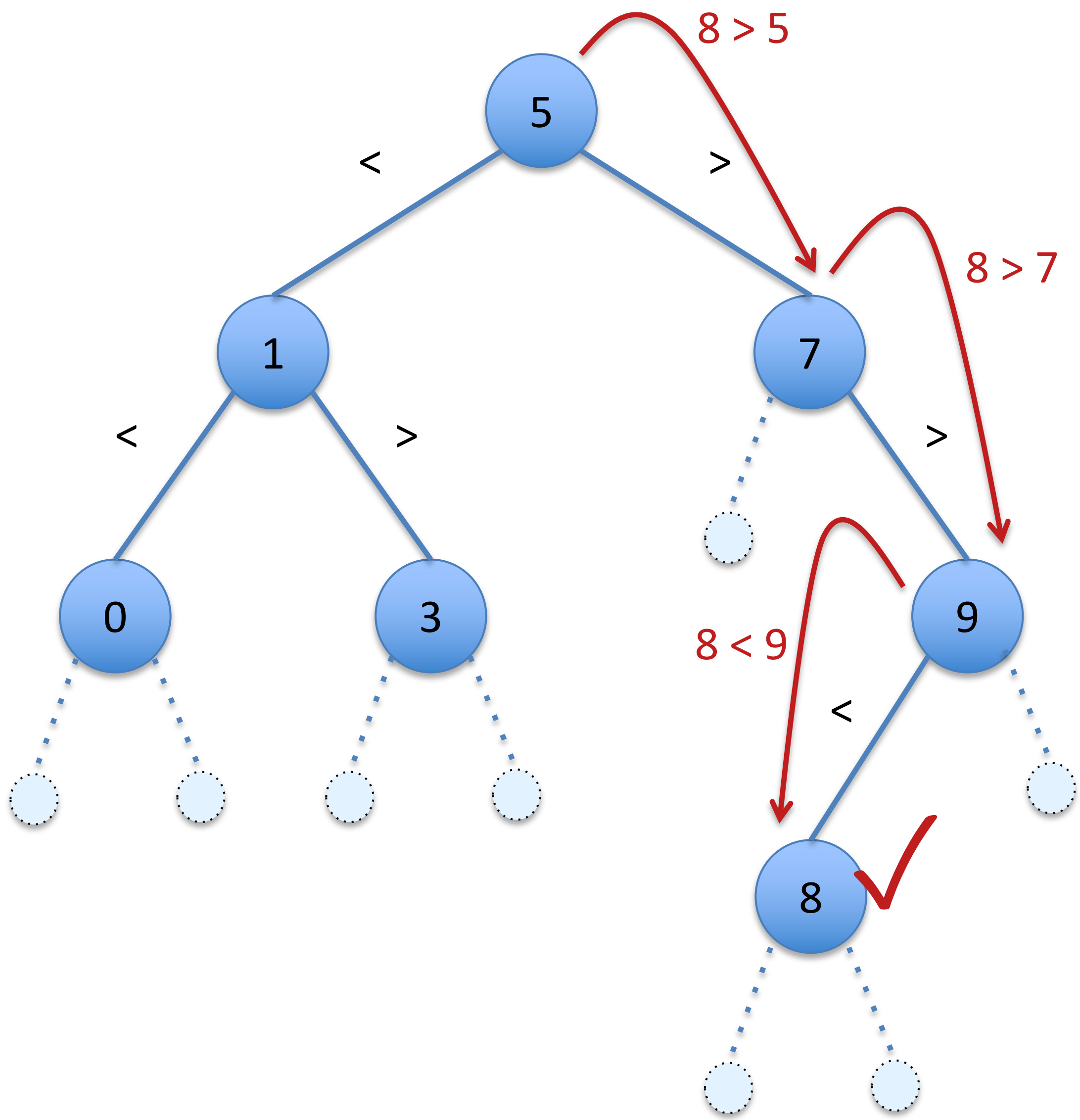
Answer: yes

Searching a BST

```
(* Assumes that t is a BST *)
let rec lookup (t:tree) (n:int) : bool =
  begin match t with
  | Empty -> false
  | Node(lt,x,rt) ->
      if x = n then true
      else if n < x then lookup lt n
      else lookup rt n
  end
```

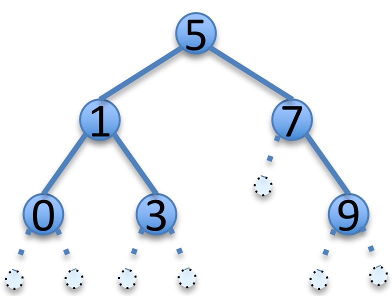
- The BST invariants guide the search.
- Note that lookup may return an incorrect answer if the input is *not* a BST!
  - This function *assumes* that the BST invariants hold of t.

Search in a BST: `(lookup t 8)`

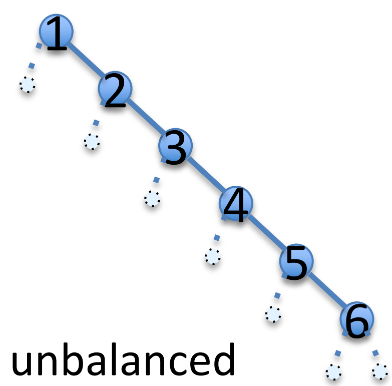


BST Performance

- `lookup` takes time proportional to the *height* of the tree.
  - not the *size* of the tree (as it did with `contains` for unordered trees)
- In a *balanced tree*, the lengths of the paths from the root to each leaf are (almost) *the same*.
  - no leaf is too far from the root
  - the height of the BST is minimized
  - the height of a balanced binary tree is roughly $\log_2(N)$ where N is the number of nodes in the tree



balanced



see bst.ml

DEMO

Manipulating BSTs

Inserting an element

insert : tree -> int -> tree

"insert t x" returns a new tree containing x
and all of the elements of t

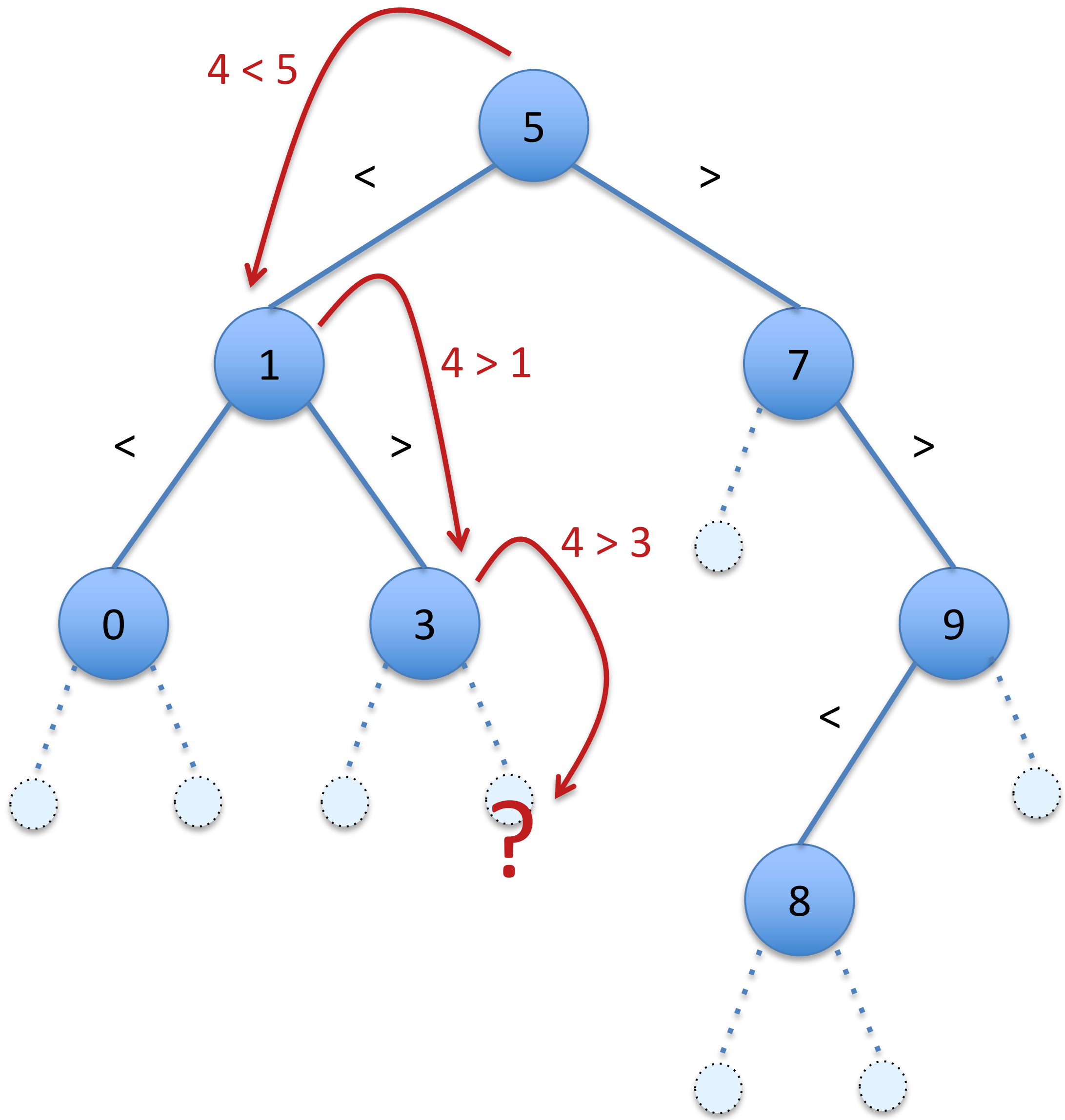
Inserting into a BST

- Challenge: can we make sure that the result of insert really is a BST?
  - i.e., the new element needs to be in the right place!

- Payoff: we can build a BST containing any set of elements
  - Starting with `Empty`, apply insert repeatedly
  - If insert *preserves* the BST invariants, then any tree we get from it will be a BST *by construction*
    - No need to check!
  - Later: we can also "rebalance" the tree to make lookup efficient (NOT in CIS 120; see CIS 121)

First step: find the right place…

Inserting a new node: `(insert t 4)`



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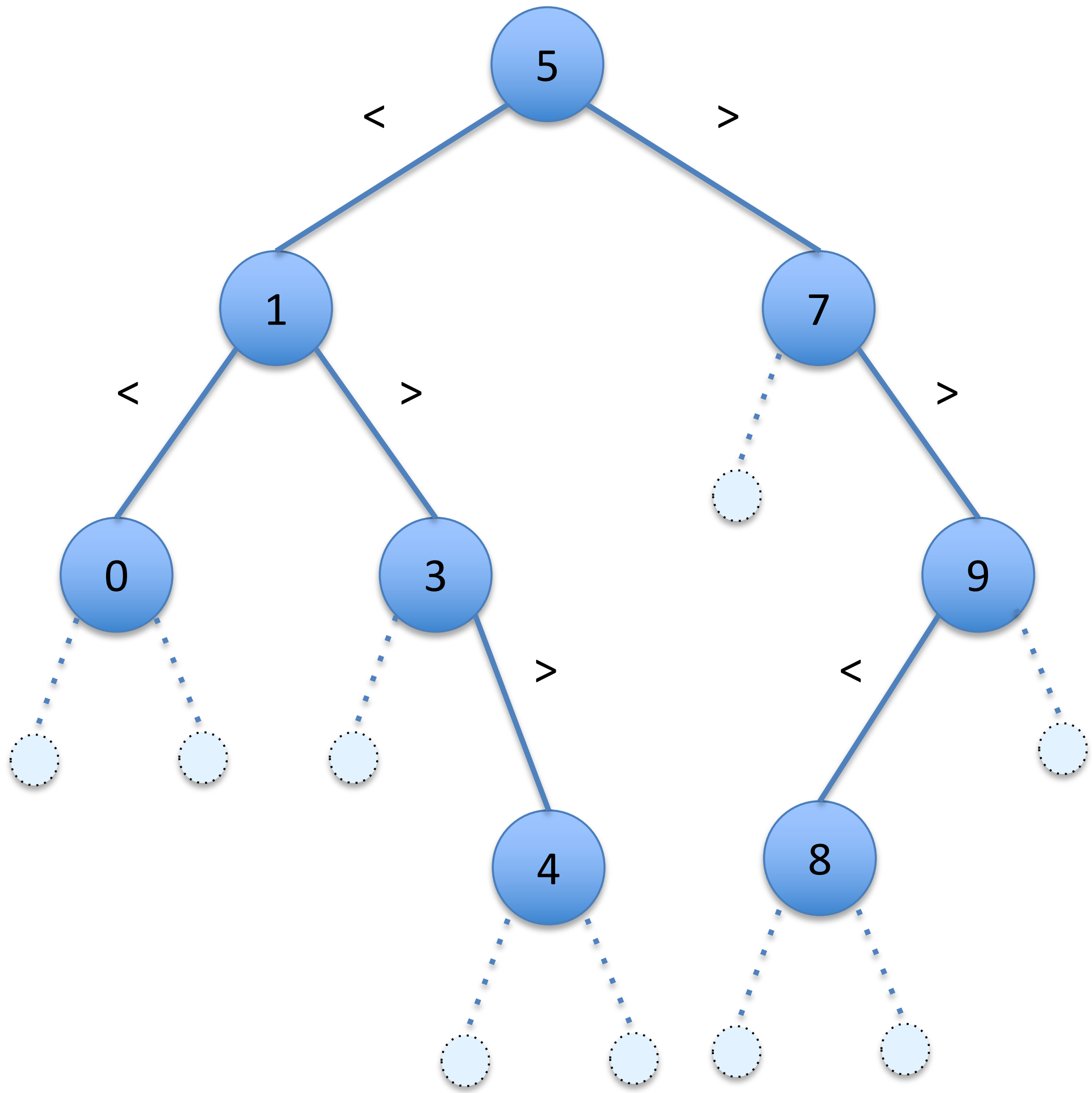


Inserting Into a BST

```
(* Insert n into the BST t *)
let rec insert (t:tree) (n:int) : tree =
  begin match t with
  | Empty -> Node(Empty,n,Empty)
  | Node(lt,x,rt) ->
     if x = n then t
     else if n < x then Node(insert lt n, x, rt)
     else Node(lt, x, insert rt n)
  end
```

- Note the similarity to searching the tree.
- Assuming that t is a BST, the result is also a BST. (Why?)
- Note that the result is a *new* tree with (possibly) one more Node; the original tree is unchanged

Critical point!

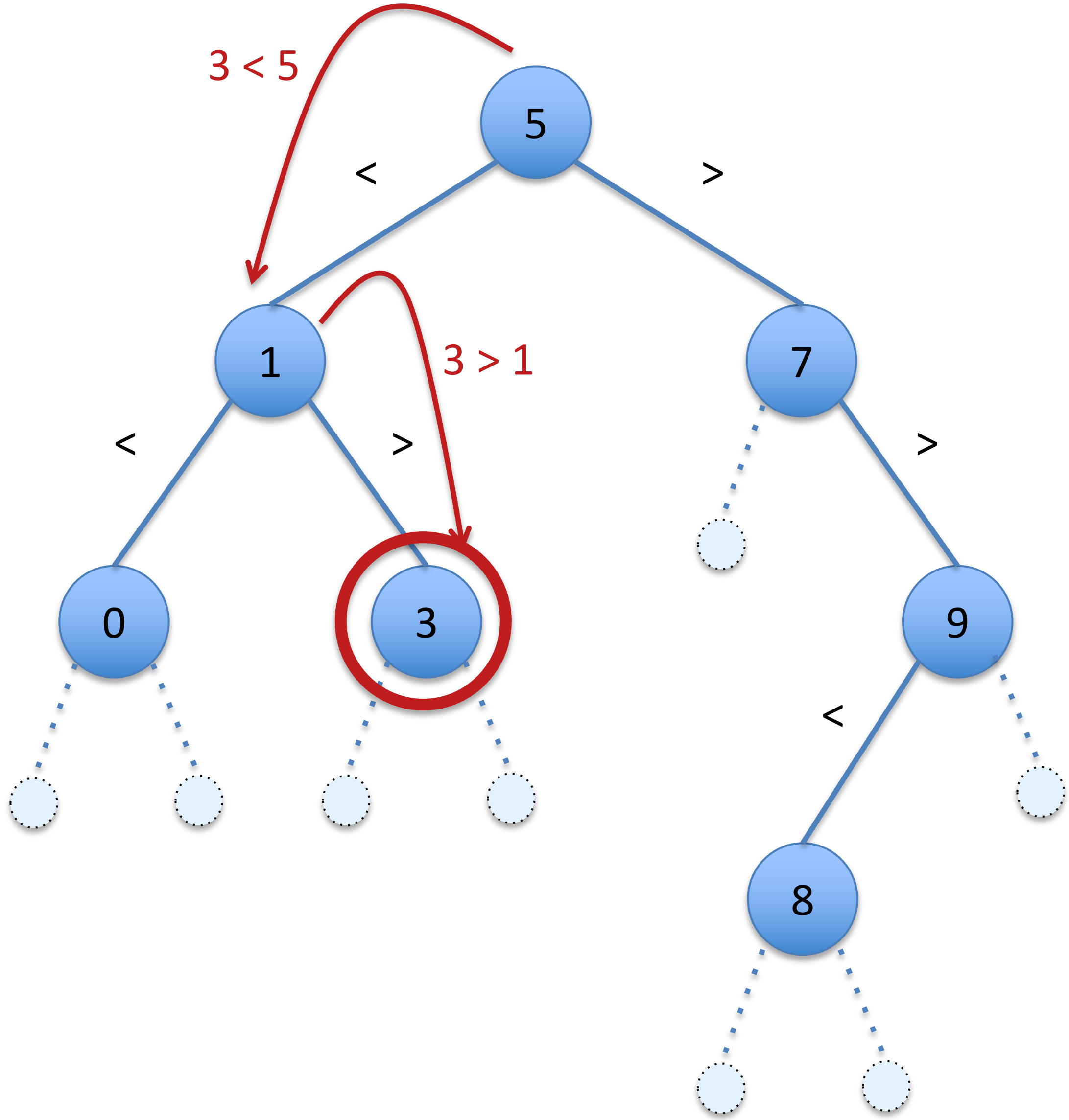
Manipulating BSTs

Deleting an element

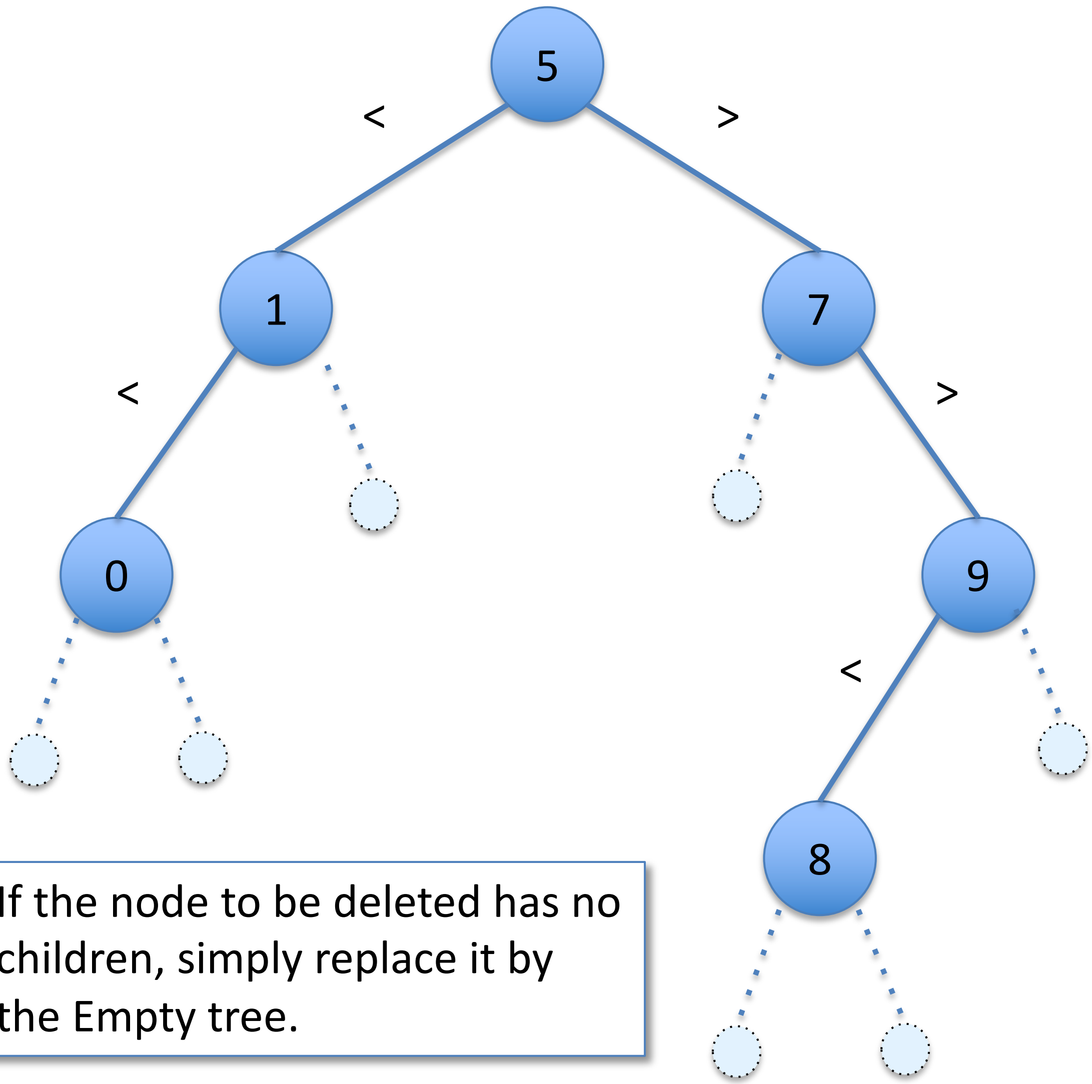
delete : tree -> int -> tree

"delete t x" returns a tree containing all of the elements of t except for x

Deletion – No Children: `(delete t 3)`

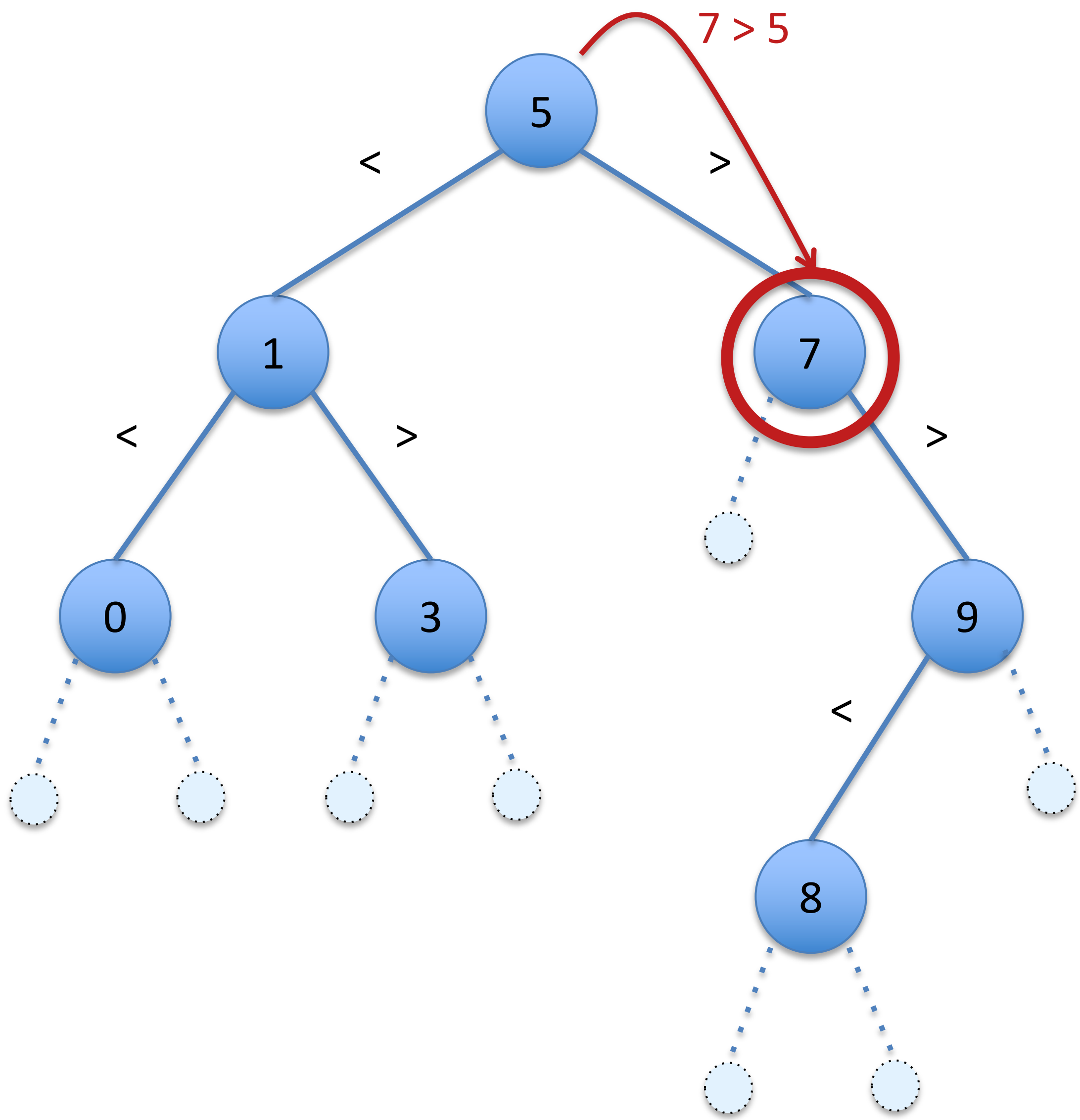


Deletion – No Children: `(delete t 3)`



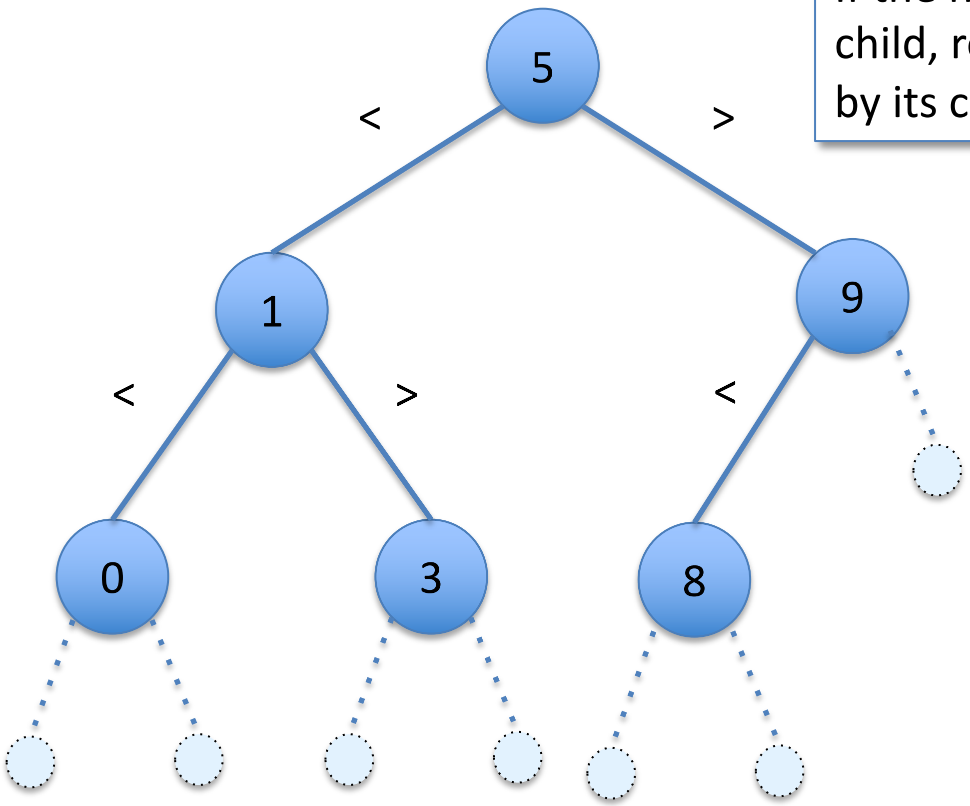
If the node to be deleted has no children, simply replace it by the Empty tree.

Deletion – One Child: (delete t 7)

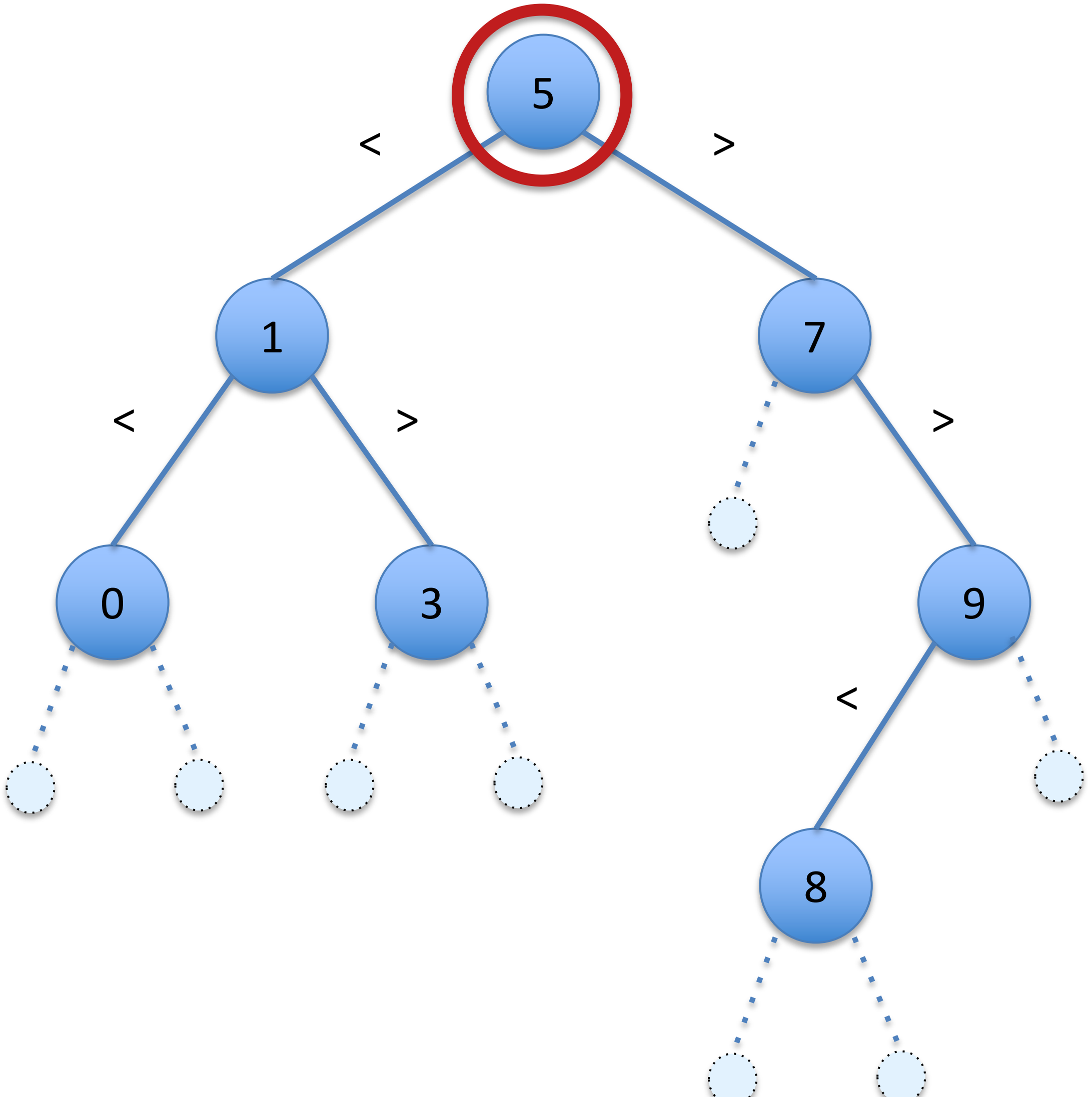


Deletion – One Child: `(delete t 7)`

If the node to be delete has one child, replace the deleted node by its child.

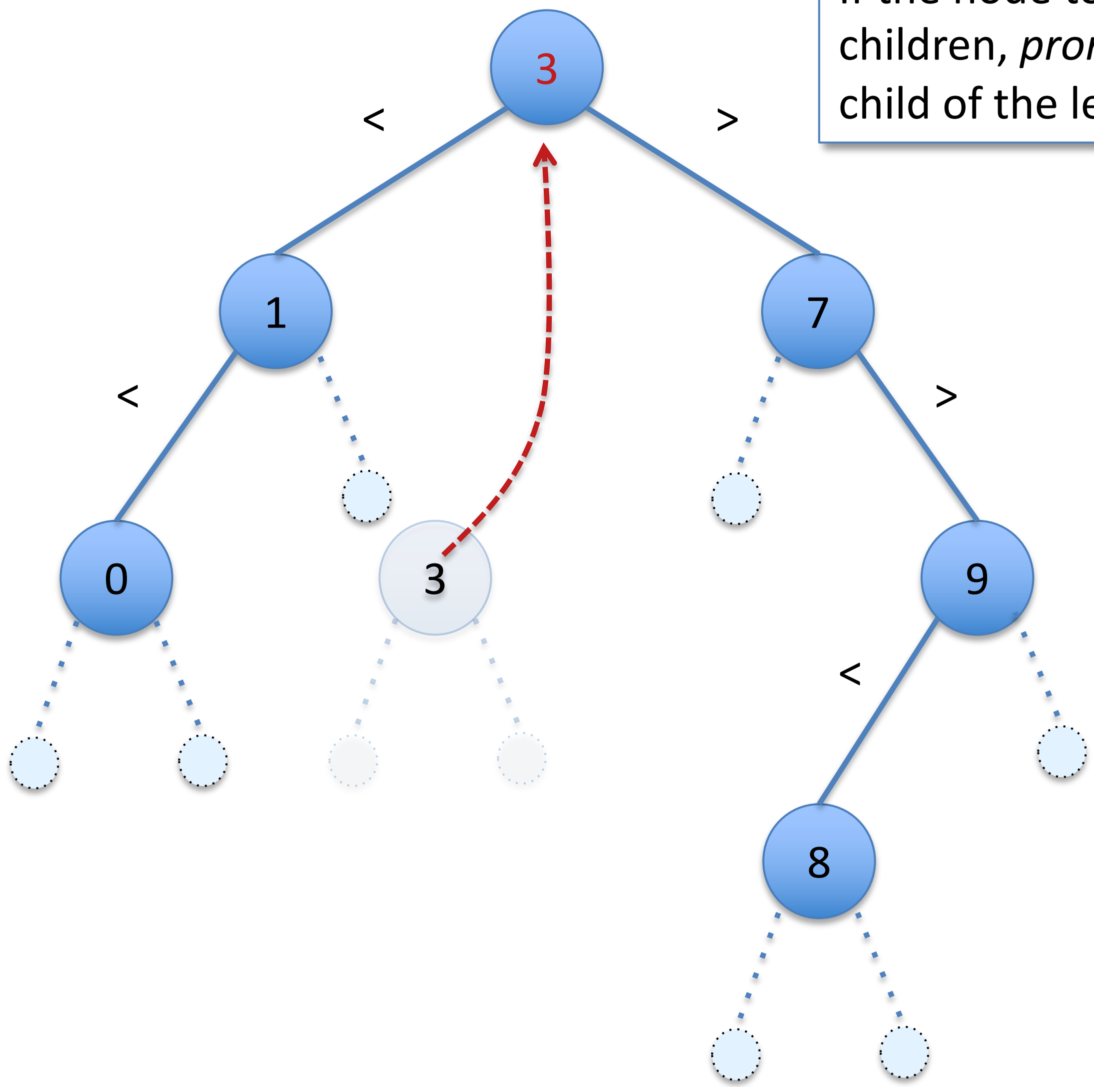


Deletion – Two Children: (delete t 5)

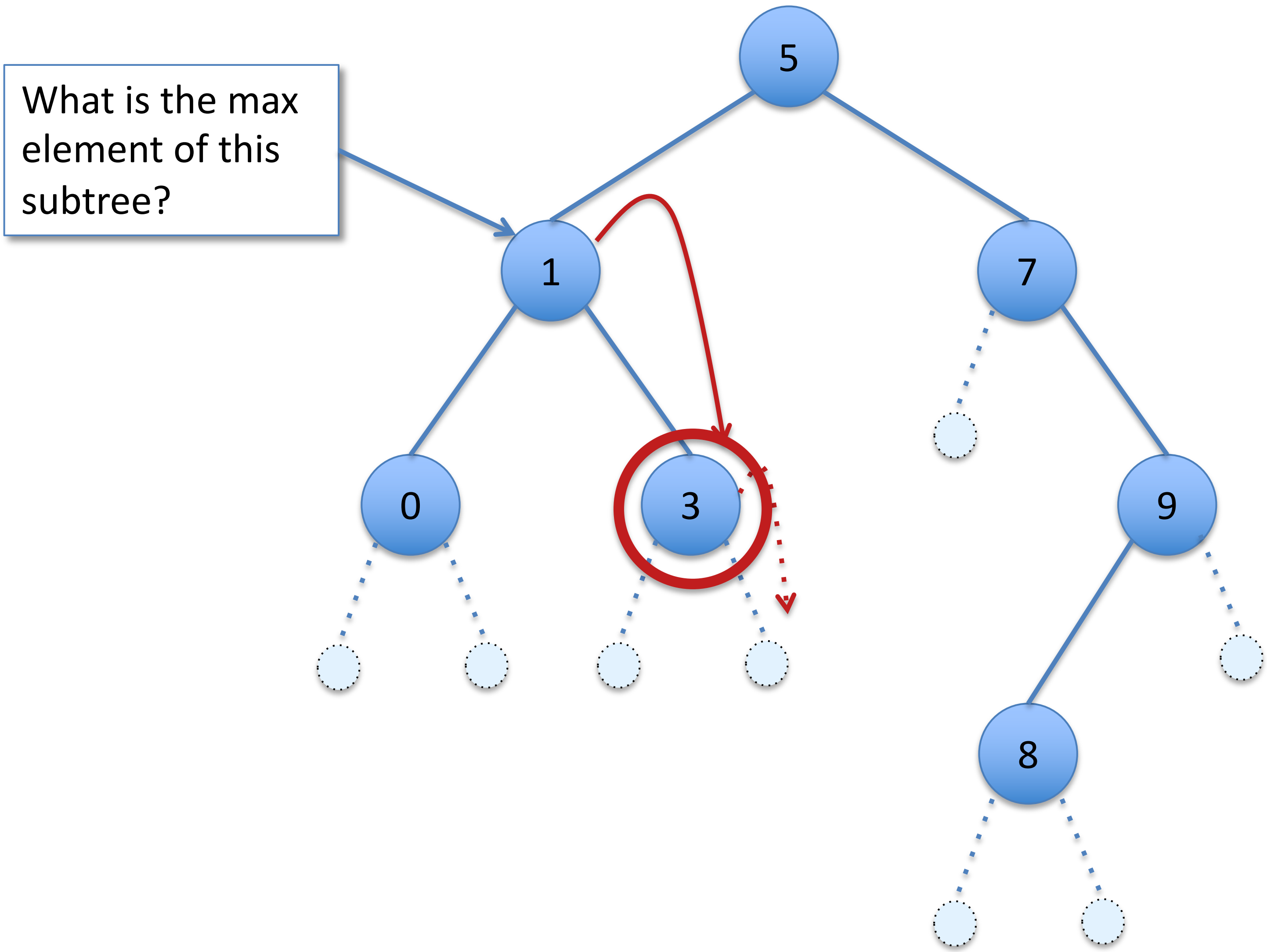


Deletion – Two Children: `(delete t 5)`

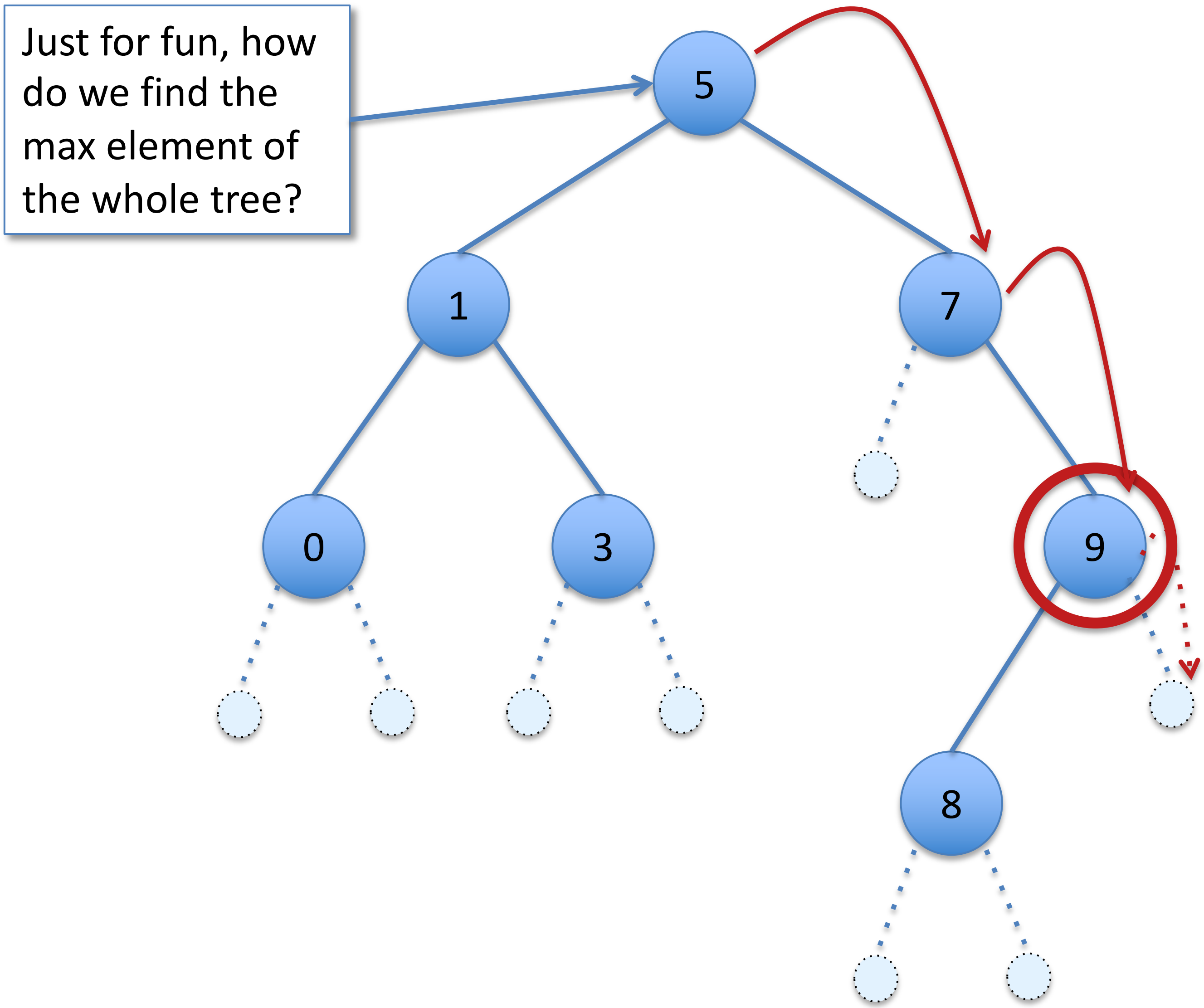
If the node to be delete has two children, *promote* the maximum child of the left tree.



How to Find the Maximum Element?



How to Find the Maximum Element?



Tree Max

```
let rec tree_max (t:tree) : int =
  begin match t with
  | Node(_,x,Empty) -> x
  | Node(_,_,rt) -> tree_max rt
  | _ -> failwith “tree_max called on Empty”
  end
```

- BST invariant guarantees that the maximum-value node is farthest to the right
- Note that `tree_max` is a *partial** function
  - Fails when called with an empty tree
- Fortunately, we never need to call tree_max on an empty tree
  - This is a consequence of the BST invariants and the case analysis done by the delete function

* Partial, in this context, means “not defined for all inputs”.

Code for BST delete

bst.ml

Deleting From a BST

```
let rec delete (t: tree) (n: int) : tree =
  begin match t with
  | Empty -> Empty
  | Node(lt, x, rt) ->
    if x = n then
      begin match (lt, rt) with
      | (Empty, Empty) -> Empty
      | (Node _, Empty) -> lt
      | (Empty, Node _) -> rt
      | _ -> let m = tree_max lt in
        Node(delete lt m, m, rt)
    end
    else if n < x then Node(delete lt n, x, rt)
    else Node(lt, x, delete rt n)
end
```

See bst.ml

Subtleties of the Two-Child Case

- Suppose Node(lt,x,rt) is to be deleted and lt and rt are both themselves nonempty trees.
- Then:
  1. There exists a maximum element, m, of lt (Why?)
  2. Every element of rt is greater than m (Why?)

- To promote m we replace the deleted node by:
  Node(delete lt m, m, rt)
  - I.e. we recursively delete m from lt and relabel the root node m
  - The resulting tree satisfies the BST invariants

If we insert a label n into a BST and then immediately delete n, do we always get back a tree of exactly the same shape?

1. yes
2. no

Answer: no (what if the node was in the tree to begin with?)

If we insert a value n into a BST *that does not already contain n* and then immediately delete n, do we always get back a tree of exactly the same shape?

1. yes
2. no

Answer: yes

If we delete n from a BST (containing n) and then immediately insert n again, do we always get back a tree of exactly the same shape?

1. yes
2. no

Answer: no (e.g., what if we delete the item at the root node?)