CIS 121 — Data Structures and Algorithms
Homework Assignment 2

Assigned:  September 11, 2018       Due:  September 17, 2018

Note:  The homework is due electronically on Gradescope and Canvas on Monday, September 17 by 11:59 pm EDT. You may NOT use any late days on this homework; solutions will be distributed in class on Tuesday to help you study for Midterm 1.

A. Gradescope: You must select the appropriate pages on Gradescope. Gradescope makes this easy for you: before you submit, it asks you to associate pages with the homework questions. Failing to do so will get you 5% off, which cannot be argued against after the fact. Gradescope may prompt you with a warning to select your cover page, please ignore this warning.

B. Canvas: You must also submit your assignment on Canvas. Forgetting to do so will incur a 10% penalty.

C. LaTeX: You must use the hw121.cls [Latex template] provided on the course website, or a 5% penalty will be incurred. Handwritten solutions or solutions not typeset in Latex will not be accepted.

D. Solutions: Please write concise and clear solutions; you will get only a partial credit for correct solutions that are either unnecessarily long or not clear. Please refer to the [Written Homework Guidelines] for all the requirements. Piazza will also contain a complete sample solution.

E. Algorithms: Whenever you present an algorithm, your answer must include 3 separate sections. Please see Piazza for an example complete solution.

1. A precise description of your algorithm in English. No pseudocode, no code.
2. Proof of correctness of your algorithm
3. Analysis of the running time complexity of your algorithm

F. Collaboration: You are allowed to discuss ideas for solving homework problems in groups of up to 3 people but you must write your solutions independently. Also, you must write on your homework the names of the people with whom you discussed. For more on the collaboration policy, please see the [course webpage].

G. Outside Resources: Finally, you are not allowed to use any material outside of the class notes and the textbook. Any violation of this policy may seriously affect your grade in the class. If you’re unsure if something violates our policy, please ask.
1. [14pts - CIS 160 Review]
   a. Let $G$ be a simple undirected graph with $n \geq 2$ vertices. Show that if $G$ is not connected, then the sum of degrees of some pair of vertices in $G$ is less than $n - 1$.

   b. You have a normal deck of 52 playing cards. Suppose you cut the deck once in a random position. Let $X$ be the number of cards that start out in the lower half before the cut and remain in the lower half after the cut.

      i. What is $E[X]$? Justify your answer.

      ii. Now suppose you cut the deck 121 times in random positions. In that case, what is $E[X]$? Justify your answer.

* To cut a deck, one takes a contiguous portion of the deck off the top and places it beneath the rest of the deck.

2. [14pts - CHOCOLATE!!!] Imagine you have an $m \times n$ chocolate bar that you want to divide into individual $1 \times 1$ pieces. You may assume $m$ and $n$ are both positive integers. Suppose some algorithm $A$ makes a series of “breaks” such that at the end of the algorithm, you are left with $mn$ individual pieces. Note $A$ can only “break” one piece at a time.

   a. Give a lower bound on the number of “breaks” $A$ makes. Justify your bound.

   b. Give an algorithm and analyze its runtime to show that this lower bound is achievable, and thus is tight.

3. [15pts - Big-Oh] Prove or disprove the following. In case of a proof, use the definitions of $O$, $\Omega$, $\Theta$ and give values of the constants in the definitions for which the conditions in the definition hold.

   a. $5n\sqrt{n} = O(\frac{1}{2}n^2 - 10)$

   b. $2^{5\lg n + \lg \lg n} \lg (n^5) = O(4^{3\lg n})$

   c. $f(n)^2 = O(g(n)^2)$ given two nonnegative functions $f$ and $g$ such that $f(n) = O(g(n))$.

4. [15pts - Recurrence Relations] For all recurrences below, assume that $T(n) = 1$ for $n \leq 10$ and $n$ is an exact power of 2. Use the method of substitution to...

   a. Prove or disprove that the solution to the recurrence $T(n) = T(\frac{n}{2}) + \log n$ is $\Theta(\log^2 n)$.

   b. Solve the recurrence, giving your answer in $\Theta$ notation: $T(n) = \sum_{i=1}^{k} T(\frac{n}{a_i}) + n$ where $a_1, a_2, ..., a_k$ are positive constants such that $\sum_{i=1}^{k} \frac{1}{a_i} \leq \frac{99}{100}$
5. [16pts - Last Minute Complications] For each of the subproblems problem below, a clear description of your algorithm and an analysis of its time complexity is sufficient. **No proof of correctness is needed.**

   a. Caroline is throwing a CIS 121 sleepover and wants to give out onesies to all the TAs. The onesies are sized by height, and Caroline realized last minute she only bought one onesie of each size. Therefore, she needs to make sure that she only has at most one TA of any given height. The TAs line up in two lines and order themselves in increasing order of height. Now, help Caroline find any TAs that have the same height as another TA and kick all but one of them (doesn’t matter who) out of the sleepover. Your algorithm should run in \(O(n)\) time, where \(n\) is the total number of TAs.

   b. After dropping out of Penn, Shirali decides to open up her own comedy club restaurant. She maintained a schedule of performers, and performers were ordered by what time they planned to arrive at the restaurant (so they wouldn’t have to wait for too long). She initially had \(n\) comedians on the schedule ready to perform, but at the last minute, \(k\) of the comedians dropped out leaving her with \(n - k\) performers. Luckily, Shirali was able to find \(k\) relatively funny friends to replace them, but they were all arriving at the restaurant at different times. Given the original sorted schedule with \(n - k\) comedians on it, and given the list of \(k\) new comedians’ arrival times, design an \(O(k \log k + n)\) algorithm that creates a lineup sorted based on each comedian’s arrival time. You may assume \(0 < k \leq n\), and that arrival times are distinct.

6. [11pts - Quicksort]

   a. Array \(A = [3, 0, 2, 4, 5, 8, 7, 6, 9]\) has just been partitioned by the first step of the Quicksort algorithm. Which of \(A\)’s elements could have been the pivot? List all possible pivots if more than one exists. Give a brief justification (1-2 sentences max) for your answer.

   b. For each sub-problem below, say how many recursive calls to Quicksort will be performed for each case (count the first one as well). You may assume that the algorithm is selecting the pivot using: Pivot = \(A[(\text{Lo} + \text{Hi}) / 2]\) and that the arrays are 0-indexed. Inputs to Quicksort are:

   (i) \(A = [1, 3, 2, 4, 5, 7, 6, 8, 9]\)
   (ii) \(B = [2, 6, 9, 5, 1, 3, 4, 7, 8]\)
   (iii) \(C = [7, 6, 8, 4, 9, 3, 5, 2, 1]\)

   c. Please show the state of the array \(A\) in (i) after each partition when Quicksort is run on it.

7. [15pts - Sarah & Co] After graduating from Penn, Sarah decided to pursue her dream of becoming a famous arbiter of fashion, and opened a high fashion clothing boutique – Sarah & Co – located on 5th Avenue, right next to Saks’ flagship store. Designers from all over the world aspired to have their latest lines sold at Sarah & Co, and Sarah & Co quickly
became THE store to discover and take part in all the latest fashion trends. Sarah’s clients include many celebrities and high-profile socialites, and she realized quickly that in order to maintain Sarah & Co’s high status, there were several services she needed to provide for her sellers and clients:

1. **addItem()** Sarah needs to be able to add new items to Sarah & Co’s collection. Items come in one at a time.

2. **sellLatestItem()** Sarah noticed that many of her customers entered the store specifically looking to buy into the latest fashion trend. To address this, Sarah decides that any customer that walks in has to buy the most recent item added to the collection.

3. **showOffMostExpensiveItem()** To “wow” her high profile clients, Sarah wants to be able to quickly show off the most expensive item currently for sale in her store. Note: she doesn’t necessarily want to remove it from her store, but just be able to show it off.

Sarah’s clients don’t want to be kept waiting, so these three operations all must be done in \(O(1)\) time. How would you advise Sarah in how she should structure her store, in order to maintain all of these operations? Please prove the correctness and runtime for all operations.

**Feedback:** How long did you spend on this assignment? What did you think of it, and how can we improve written assignments in the future? Let us know anonymously at: [https://tinyurl.com/CIS121-18fa-hw-feedback](https://tinyurl.com/CIS121-18fa-hw-feedback)

We really appreciate your feedback!