OH TODAY: 1:15 pm - 2:15 pm

Final Exam: please see classpage/Plazza for date/time & other details.

**Skip list**

**Lemma:** The height of a skip list with \( n \) elements is \( \leq 2 \log_{1/n} n \) with high probability \((1 - \frac{1}{n})\).

**Lemma** For any int \( r \) s.t. \( 0 < r \leq n \),

\[
\sum_{i=0}^{r} (\frac{n}{r})^i \leq \left( \frac{ne}{r} \right)^r
\]

**Thus:** With a high probability, every search in a skip list with \( n \) elements
takes $O(\log n)$ time.

**Proof:** When we search for element $x$,
- we start from the top and follow down links and right links until we find the largest element $\leq x$ in $L_i$. 
Search time: # Down links + # right links.
Instead, we will start at Vi and retrace the search path in the reverse direction.
Thus we go up or left.

# Search time: # Up links + # left links.

We already know that the hit of the skip list with n elements \( \leq 2 \log n \),
with a high prob. (i.e., with a
\[ \text{prob of } \geq \left( -\frac{1}{n} \right). \text{ In other words,} \\
\text{hit of the skip list } > 2\log n \text{ with a} \\
\text{prob of } \leq \frac{1}{n}. \\
\]

\[ \therefore \# \text{HEADS (\# up links)} > 2\log n \text{ with a} \\
\text{prob } \leq \frac{1}{n}. \]

We will prove that the total \# coin 
\text{tosses } \leq 20\log n \text{ with a high prob.} 
That is, after we toss 20 \log n \text{ coin} 
\text{tosses, the prob that our search} 
\text{has not ended } \leq \frac{1}{n}. 

E: “bad” event that at the end of
201gn coin tosses our search hasn't ended.

\( U \): event that \# Heads > 21gn.

\( \Pr [E] \leq \) ?

\[
\Pr [E] = \Pr [E \land U] + \Pr [E \land \overline{U}]
\]

\[
\leq \Pr [U] + \Pr [E] \cdot \Pr [\overline{U} \mid E]
\]

\[
\leq \frac{1}{n} + \left( \Pr [\overline{U} \mid E] \right)
\]

\[
\Pr [\overline{U} \mid E] = \sum_{i=0}^{21gn} \binom{201gn}{i} \left( \frac{1}{2} \right)^i \left( \frac{1}{2} \right)^{201gn-i}
\]

\[
= \left( \frac{1}{2} \right)^{201gn} \sum_{i=0}^{21gn} \binom{201gn}{i}
\]

\[
= \frac{201gn}{21gn} \sum_{i=0}^{21gn} \binom{201gn}{i}
\]

\[
\text{to}
\]

\[
-201gn \cdot \ln \left( \frac{201gn}{21gn} \right)
\]
\[
\leq 2 \left( \frac{-2\ln n}{2\ln n} \right) \\
= 2^{-2\ln n} \cdot \left( \ln \left( 10e \right) \right)^{2\ln n} \\
= 2^{-2\ln n} \cdot \left( \frac{\ln 10e}{2} \right)^{2\ln n} \\
= 2^{-2\ln n} \cdot \left( \ln 10e - 1 \right)^{2\ln n} < -1 \\
= 2^{\frac{1}{2\ln n}} \cdot \sqrt{\frac{1}{n^2}}.
\]

\[
\Pr(E) \leq \frac{1}{n} + \frac{1}{n^2}.
\]

\[
\leq \frac{1}{n^2} + \frac{1}{n} = \sqrt{\frac{2}{n}}.
\]
Min cut.

Input: Undirected graph $G = (V, E)$.

Obj: To partition $V$ into $V_1$ and $V_2$ such that the number of edges crossing $V_1$ and $V_2$ is minimized.

Karger's alg.

- Pick an edge uniformly at random.
- Contract it.
- Repeat until we have 2 components left.
Analysis.

$C$: my favorite min-cut.

$|C| = k$.

$E_i$: event that cut $C$ is "safe" in iteration $i$.

$$P_r[E_1 \cap E_2 \cap \ldots \cap E_{n-2}]$$

$$= P_r[E_1] \cdot P_r[E_2 | E_1] \cdot P_r[E_3 | E_1 \cap E_2] \ldots$$

$P_r[E_1] = 1 - P_r[\bar{E}_1]$

$m = \frac{\sum_{w \in V} d(w)}{2} = 1 - \left( \frac{k}{m} \right)$

$\geq \frac{n \cdot k}{2} \geq 1 - \frac{1}{n \cdot k / 2}$

$m \geq \frac{n \cdot k}{2} \left( \frac{m}{\sum_{w \in V} d(w)} \right)$

$\geq \left( \frac{m^2}{\sum_{w \in V} d(w)} \right) = \left( \frac{m^2}{\sum_{w \in V} d(w)} \right)$
\[ P(X_k = 1 - \frac{k}{m_1} \geq 1 - \frac{k}{(n-1)^{\frac{1}{2}}} P_1 \] Similarly \[ P(E_3 | E_1 \cap \neg E_2) = 1 - P(E_3 | E_1 \cap \neg E_2) = 1 - \frac{k}{m-2} \]

\[ P(E_2 | E_1) \geq 1 - \frac{2}{n-1} \]

Similarly, \[ P([E_3] E_1 \cap \neg E_2) \geq 1 - \frac{2}{n-2} \]

\[ P(E) \geq \left( 1 - \frac{2}{n} \right) \left( 1 - \frac{2}{n-1} \right) \left( 1 - \frac{2}{n-2} \right) \cdots \left( \frac{3}{5} \right) \left( \frac{2}{4} \right)^2 \]
\[ \left( \frac{n-2}{n} \right) \left( \frac{n-3}{n-1} \right) \cdots \left( \frac{2}{3} \right) \left( \frac{1}{2} \right) \left( \frac{1}{2} \right) \]

\[ \frac{2}{n(n-1)} = \frac{1}{\binom{n}{2}} \]

\[ \binom{k}{p} \binom{k-1}{p} \binom{k-2}{p} \cdots \binom{1}{p} \]

Smallest \ 0/p.

\[ \left( 1 - \frac{2}{n(n-1)} \right)^p \leq e^{-\frac{2}{n(n-1)}p} \]

Choose \( p = \frac{n(n-1)}{2} \cdot \ln n \)

\[ \Pr[\text{bad event}] \leq e^{-\ln n} = e^{-\frac{1}{n}}. \]
Stoer-Wagner.

Min cut: (Steiner model).

\[ e_1, e_2, \ldots, e_m \]

\[ \Rightarrow e_1 \quad \cdots \quad e_m \]

Input: Boolean 2-SAT formula.

\[ (x_1 \lor \overline{x_2}) \land (x_2 \lor \overline{x_3}) \land (x_2 \lor \overline{x_4}) \land (x_3 \lor x_4) \]

Clause

\[ \ldots \]

\[ x_1, x_2, \ldots, x_n : \text{boolean variables.} \]

/ 

T/L
To assign truth values to $x_1, x_2, \ldots, x_y$ s.t. $\phi$ is satisfiable or output it is not satisfiable.

1. Start with an arbitrary truth assignment.

2. If $\phi$ is satisfiable then
   
   return yes

3. else
   
   - pick an unsatisfied clause.
   
   - choose a literal uniformly at random.
   
   - flip its value.

4. Repeat until find.
\[ \sim n^1 \text{ iterates.} \]