NO OH TODAY.

**Breadth First Search (BFS)**

- Layu by layu.
- **L₀**: S
- **L₁**: Consists of all neighbors of S.
- **L₂**: " " " " " neighbors of S, which have not yet been discovered.
- **Lₙ**: Consists of all vertices that are neighbors of vertices in Lₙ-1.
Lemma: The output of BFS is a tree (rooted at S).

Lemma: Consider a vertex at layer Li. Can we say anything about \( \text{dist}(s,i) \)?

\[
\text{min } \# \text{edges to go from } s \text{ to } i \text{ in } G?
\]

Ans: \( j \).

Lemma: Let \( T \) be a BFS tree. Let \( u \) and \( v \) be vertices in \( G \) s.t. \( (u,v) \in E \).

Let \( u \in L_i \) and \( v \in L_j \). Then \( |i-j| \leq 1 \).

Proof Sketch: WLOG, let \( i < j \). Note that \( j = i + 1 \).
Graph Representation.

Adj. Matrix:

\[ M_{i,j} = \begin{cases} 1 & \text{if } (i,j) \in E \\ 0 & \text{otherwise} \end{cases} \]

Adj. List representation.
(i) \((u,v) \in E\)?

\[ O(1) \rightarrow \text{Adj Matrix} \]
\[ O(\deg(u)) \rightarrow \text{Adj list} \]

(ii) Space complexity?

\[ \Theta(n^2) - \text{Adj Matrix} \]
\[ O(n^2) - \text{Adj list} \]
\[ \Theta(n+m) \left( \Theta(n + \sum_{u \in V} \deg(u)) \right) = \Theta(n + 2m) \geq \Theta(n+m) \]

(iii) Suppose we are at vertex \(u \in V\). We want to say “hi” to all of \(u\)'s neighbors.
Adjacent Matrix: $O(n^2), \Theta(n^2)$
Adjacent List: $\Theta(d_{avg}(n))$

$O(1)$ per neighbor.

Unless specified otherwise, you may assume Adj. list representation.

BFS ($G, s$):

for each $u \in V$ do

  discovered ($u$) $\leftarrow$ False

  discovered ($s$) $\leftarrow$ True

  $L[0] \leftarrow \{s\}$

  $i \leftarrow 0$

  $T \leftarrow \varnothing$

while $L[i] \neq \varnothing$ do

  $L[i+1] \leftarrow \varnothing$

  for each $u \in L[i]$ do

    discovered ($u$) $\leftarrow$ True

  $i \leftarrow i + 1$

end while

$\Theta(n)$

$O(1)$

can be implemented as Queue.
At that time, we go to each $v_i$ at most once.

Each vertex in $G$ appears in a long loop at most once.

```
while (loop $L$ is True)
  for each neighbor $w$ of $v_i$
    if $w$ is known
      then $L[ci] = L[ci] U \{w\}$
    else $L[ci] \leftarrow \{w\}$
    $i \leftarrow i + 1$
```

For each vertex $v_i$ of $G$, do

```
T \leftarrow 0
for $i = 1, \ldots, n$
  return $T$
```
neighbors and $\deg(u)$ work.

- Amount of time per vertex: $\Theta(\deg(u))$.

- Total time of the while loop

\[ \sum_{u \in V} \deg(u) = O(n + m) \]

- Total runtime of BFS: $O(n + m)$.

\( n \): \# vertices in \( G \), \( m \): \# edges in \( G \).

Problem (Testing bipartiteness)

**Input:** Undirected graph \( G = (V, E) \)

**Obj:** Output Yes, if bipartite. No, or on 2-colorable graphs.
AHF: \[\text{for each } e = (u,v) \in G, \]
\[\text{check if } u \text{ and } v \text{ belong to different partition. If this is true for each } e \text{ then o/p Yes, else o/p No.} \]

1. \( T \leftarrow \text{BFS}(G) \)
\[\text{L} \rightarrow O(n+m) \]

2. Color vertices in even layers Blue and vertices in odd layers Red. \(\rightarrow O(n) \)

3. If any edge \((u,v) \in G\) has both its end pts of the same color then o/p No.

4. else o/p Yes.

\[O(m)\]

\[O(n+m) \]
If \((u,v) \in E\) and \(u \& v\) belong to the same layer then we have an odd-length cycle (with \(s\)).

Both \(u \& v\) are at the same dist from \(s\) (lowest common ancestor of \((u,v)\)).

\[ \therefore \quad \text{dist}(s,u) + 1 \]

**Correctness.**

If our alg o/p is yes then we have a 2-colorable graph & of course,
$G$ is bipartite.

If our alg. outputs No, we need to argue that $G$ is not 2-colorable. We can show this by proving that $G$ has an odd-length cycle.

Our alg. outputs No, if there is an edge $(u,v)$ s.t. $u$ & $v$ are colored the same.

Case I: $u$ & $v$ belong to different layers.

Not possible (Lemma)

Case II: $u$ & $v$ belong to the same layer.
We argued that $G$ must contain an odd length cycle formed with $u, v \land \text{lca}(u, v) \rightarrow \text{lowest common ancestor}.$