# CIS 121
## Practice Problems for Exam 1
### February 11, 2018

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**1.** Prove using induction that $n$ is $O(2^n)$.

**2.** Prove that $2^{(n^2)}$ is not $O(5^n)$. Do *not* use any theorems about Big-Oh that you might happen to know other than the definitions.

**3.** Solve the following recurrence. Give a tight bound, i.e., express your answer using the $\Theta$ notation. Assume that $T(n) = 1$, when $n = 1$.

$$T(n) = T(n-1) + 1/n$$

**4.** Consider the following code fragment

```
for(int i=1;i<=n;i=2*i)
  for (int k = i; k >0; k = k/2)
    print('*');
```

**a.** Compute the number of stars printed as a function of $n$. You can assume that $n = 2^m$.
**b.** Give a $\Theta$-bound.

**5.** Modify the quicksort algorithm so that its running time is $O(n \log n)$ in the worst case. You may assume that all elements are distinct.

**6.** Suppose that you have a "black box" worst-case linear-time median subroutine. Give a simple, linear-time algorithm that solves the selection problem for any arbitrary order statistic, i.e., given an unsorted array $A$ containing $n$ elements and an integer $i$, in $O(n)$ time, your algorithm should return the element in $A$, which is the $i^{th}$ smallest element in $A$. Your algorithm must use the black-box median-finding subroutine. You may assume that $i$ lies within the bounds of the input array $A$ and that $n$ is a power of 2. Justify your answer.

# Some Useful Facts

1. $\lg n = \log_2 n$
   $\ln n = \log_e n$

2. Below are some formulas that may come handy.

   - $\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$
   - $\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$
   - $\sum_{i=1}^{n} i^3 = \frac{n^2(n+1)^2}{4}$
   - $\sum_{i=0}^{n} c^i = \frac{c^{n+1}-1}{c-1}, c \neq 1$
   - $\sum_{i=0}^{\infty} c^i = \frac{1}{1-c}, |c| < 1$
   - $\sum_{i=1}^{\infty} c^i = \frac{c}{1-c}, |c| < 1$
   - $\sum_{i=0}^{\infty} ic^i = \frac{c}{(1-c)^2}, |c| < 1$
   - $H_n = \sum_{i=1}^{n} \frac{1}{i} = \ln n + O(1)$

3. **Simplified Master Theorem.** Let $a \geq 1$, $b > 1$ be constants and let $T(n)$ be the recurrence

   $$T(n) = aT\left(\frac{n}{b}\right) + \Theta(n^k)$$

   defined for $n \geq 0$ (we assume that $n$ is a power of $b$, though this does not make a difference in asymptotic analysis). The base case, $T(1)$ can be any constant value. Then
   **Case 1:** if $a > b^k$, then $T(n) \in \Theta(n^{\log_b a})$.
   **Case 2:** if $a = b^k$, then $T(n) \in \Theta(n^k \log_b n)$.
   **Case 3:** if $a < b^k$, then $T(n) \in \Theta(n^k)$.