On Monday, June 17, we will have our first exam from 10AM to 12PM. The exam will take place in DRL A8, and last for 120 minutes. Please be in DRL A8 a few minutes early so we have time to seat everybody properly.

This is an exam review document with readings, a mock (practice) exam, and more practice problems. You should solve the practice exam while timing yourself.

Solutions to the practice exam will be posted Saturday June 15, in the afternoon.

More information about review sessions on Sunday will follow.

1 Readings

STUDY IN-DEPTH... ...the posted notes for lectures 1-8.

STUDY IN-DEPTH... ...the posted guides for recitations 1-3. (The recitation guide for Week 3 will be posted after recitation on Thursday.)

STUDY IN-DEPTH... ...the solutions to homework 1-3. Compare with your own solutions. (Homework 3 solutions will be available on Saturday and Sunday at office hours.)

STUDY IN-DEPTH... ...the solutions to the mock exam and the additional problems contained in this document, to be posted Saturday June 15, in the afternoon. Until then, try very hard to solve these on your own.

2 Mock Exam (120 minutes for 240 points)

1. (50 pts)

For each statement below, decide whether it is TRUE or FALSE and circle the right one. In each case attach a very brief explanation of your answer.

(a) \( \binom{100}{51} \) is strictly bigger than \( \binom{100}{49} \).

(b) In Pascal’s Triangle, there exist four binomial coefficients \( c_1, c_2, c_3, c_4 \) such that \( c_1 = c_2 + c_3 + c_4 \).

(c) The boolean expressions \( \neg[(p \Rightarrow q) \lor q] \) and \( p \land \neg q \) are logically equivalent.

(d) Let \( A \) be a finite set. All functions \( f : A \to A \) are bijections.

(e) Let \( A, B \) be finite set. Then, there are exactly as many subsets of \( A \times B \) as there are functions with domain \( A \) and codomain \( 2^B \).

(f) Assume that \( B \) is a set with 7 elements and that \( A \) is a set with 15 elements. Then, for any function \( f : A \to B \) there exist at least 3 distinct elements of \( A \) that are mapped by \( f \) to the same element of \( B \).
(g) For any $A, B, C$ non-empty finite sets, let $m = |A| + |B| + |C| - |A \cup B \cup C|$ and $n = |A \cap B| + |B \cap C| + |C \cap A|$. Then $m > n$.

(h) If the set $A$ has $n$ elements then there are $n!$ injective functions with domain $A$ and codomain $A$.

(i) The Fibonacci number $F_{100}$ is even.

2. (20pts)
Count the number of distinct sequences of bits (0’s, 1s) of length 101 such that:
- there are 3 more 1’s than 0’s in the sequence; and...
- ...also the middle bit is a 1.

3. (20pts)
Prove that for any $x, y, z \in \mathbb{Z}$ such that $x + 2y = z$, if $z - x$ is not divisible by 4 then $x + y + z$ is odd.

4. (20pts)
Let $A, B$ be any sets such that $A \cap \{1, 2\} = B \cap \{1, 2\}$. Prove that the sets $(A \setminus B) \cup (B \setminus A)$ and $\{1, 2\}$ are disjoint.

5. (20pts)
Let $n \in \mathbb{N}$ and $n \geq 3$. Give a combinatorial proof (no other kinds of proofs will be accepted) for the following identity

$$\binom{n+2}{3} = \binom{n}{1} + 2 \binom{n}{2} + \binom{n}{3}$$

6. (20pts) My 6th grade teacher of Russian was unable to pay attention to what we were answering and it appeared to us that he was assigning grades completely randomly. Let’s assume that his grading rubric consisted of tossing a fair coin six times, counting the number $k$ of heads and assigning the grade $4 + k$ (our grades were in the 1-10 range).

(a) What was the probability that I would get a 10?

(b) What was the probability of the following event: “my grade was divisible by 4 or (non-exclusive or!) it was bigger than or equal to Lady Gaga’s shoe size (a 6)”?

7. (25pts)
Let

$$R_n = \sum_{k=1}^{2n} (-1)^{k+1}k$$

for $n \geq 1$.

(a) Compute $R_1, R_2, R_3$. Guess a simple way to express $R_n$ in terms of $n$. Prove your guess by induction.

(b) Prove by induction that for all $n \geq 1$ we have

$$1 + 3 + 5 + \cdots + (2n-1) = n^2$$
(c) Use the identity in part (b) and other identities that you were supposed to memorize to prove the identity in part (a).

8. (20pts)
Alice has a strange coin that shows the number 3 on one side and the number 5 on the other. Still, the coin is fair. Bob has strange die that shows the numbers 5,6,7,8,9,10 on its six faces. Still, the die is fair. Alice flips the coin and, independently, Bob rolls the die. What is the probability that the number on the die is divisible by the number on the coin?

9. (15pts)
A lottery urn contains \( n \geq 2 \) distinct balls labeled with the numbers 1,\ldots,\( n \). You extract two distinct balls from the urn. Suppose they are labeled \( i \) and \( j \). You compute \( i+j \) and write down the answer on a piece of paper. Then you put the two balls back. You repeat this \( m \) times. What is the smallest value of \( m \) that ensures (no probabilities in this problem!) that you will end up writing the same number at least twice on the piece of paper. Prove your answer.

10. (10pts)
Let \( X \) be a nonempty finite set. Consider the set \( W = \{(A, B) \mid A, B \in 2^X \land A \subseteq B\} \). Prove that \( W \) has exactly as many elements as there are functions with domain \( X \) and codomain \{1,2,3\}.

3 Additional Problems

1. For each statement below, decide whether it is TRUE or FALSE and circle the right one. In each case attach a very brief explanation of your answer.

(a) There are exactly three surjective functions with domain \{1,2\} and codomain \{a,b\}.
(b) Exactly two of the following three boolean expressions: \( p \Rightarrow q, p \land \neg q \), and \( \neg p \lor q \) are logically equivalent.
(c) Let \( A \) be a finite set. For any function \( f : A \rightarrow A \) we have \( |\text{Ran}(f)| = |A| \).
(d) There exist two distinct functions with domain and codomain \{a,b\} that are their own inverses.
(e) For any two finite sets \( A, B \), \( |2^{A \times B}| > |2^A \times 2^B| \).
(f) Recall that for any \( n = 0,1,2,3,\ldots \) row \( n \) of the Pascal Triangle contains the binomial coefficients of the form \( \binom{n}{k} \) for \( k = 0,1,\ldots, n \). \( \binom{7}{4} \) can be expressed as a sum of binomial coefficients from row 5.

2. Give a boolean expression \( e \) with three variables \( p, q, r \) such that \( e \) has the following properties:

- \( e = T \) when \( p = q = T \) and \( r = F \), AND
- \( e = F \) when \( p = F \) and \( q = r = T \).
Also, construct a truth table for e. Make sure to include all intermediate propositions as a separate column. (Yes, there are many possible answers.)

3. In how many different ways can we arrange all the letters from the English alphabet (26 characters) in a sequence such that:
   - each letter occurs exactly once, AND
   - the 5 vowels (a,e,i,o,u) occur in 5 consecutive positions.

4. Give combinatorial proofs (no other kind of proofs will be accepted) for the following identities:

   (a) \( \binom{n}{r} \binom{r}{k} = \binom{n-k}{r-k} \) (where \( k \leq r \leq n \))

   (b) \( \sum_{i=k}^n \binom{i}{k} = \binom{n+1}{k+1} \) (where \( k \leq n \))

5. Prove that for any integers \( a, b \in \mathbb{Z} \) we have \( a^2 - 4b \neq 2 \).

6. Consider \( n \) (distinguishable) bins labeled \( B_1, \ldots, B_n \) and \( r \) indistinguishable (identical) marbles. We wish to put the \( r \) marbles into the \( n \) bins in such a way that each bin will contain at least one marble and at least three of the bins will contain two or more marbles. Assume \( r \geq n + 3 \). In how many different ways can this be done?

7. For each statement below, decide whether it is TRUE or FALSE. In each case attach a very brief explanation of your answer.

   (a) The word QWERTY has 6! anagrams. (Recall that a word is a valid anagram of itself.)

   (b) The contrapositive of \( p \rightarrow q \) is logically equivalent to \( p \land \neg q \).

   (c) For any \( 2 \leq k < n \), if \( A \) has \( n \) elements then the number of subsets of \( A \) of \( k \) elements is \( \frac{n!}{(n-k)!} \).

   (d) If the set \( A \) has \( n \) elements then there are \( n! \) injective functions with domain \( A \) and codomain \( A \).

   (e) There is no set \( X \) such that \( 2^X = \emptyset \).

8. The Taney Dragons are going to the Little League World Series! In appreciation, each of the 12 distinct team members (players) can pick 2 hats from a supply of red (Philly Phillies), blue (Boston Red Sox), and green (Ploiesti Frackers) hats. For each color, the supply is unlimited. For each of the three questions below (see also next page), give the answer and an explanation of how you derived it. No proofs required.

In how many different ways can the hat picking be done if:

   (a) There is no ordering among the two hats that each player picks, and both hats can even be of the same color.
(b) The ordering matters and the two hats have a different color: let’s say each player picks a hat to wear in the morning and then a hat (of a different color) to wear in the afternoon.

(c) What is the count for part (8a) above, if you also know that at least one of the hats that Dragon’s pitcher Mo’ne Davis picks is red.

9. Recall from homework that the boolean expression $e_2$ is a **logical consequence** of the boolean expression $e_1$ if every truth assignment to the variables that makes $e_1$ true also makes $e_2$ true.

Let $x, y$ be arbitrary boolean variables. Prove, using truth tables, that $x \rightarrow y$ is a logical consequence of $\neg x \land y$.

10. Provide examples for the following. You do not have to prove that they work.

(a) For arbitrary $n \geq 1$, give an example of a set $Y$ and a function $f : [1..n] \rightarrow Y$ such that $f$ is injective but not surjective.

(b) For arbitrary $n \geq 2$, give an example of a set $X$ and a function $g : X \rightarrow [1..n]$ that is not injective and moreover $|\text{Ran}(g)| = n - 1$.

11. Punch happily tells Judy that he proved two new theorems and he shares his proofs with her.

(a) **Punch’s First Theorem:** If $n$ is odd then $n^2 - 1$ is a multiple of 4.

**Punch’s Proof:** “We prove the contrapositive instead. Suppose $n$ is even, then $n^2$ is even, then $n^2 - 1$ is odd so it cannot be a multiple of 4. Done.” Upon reading these, Judy remarks that while the theorem is true, the proof is not proving the theorem, but another statement, which is not the contrapositive of the theorem.

i. What is the contrapositive of the theorem and what statement is Punch actually proving?

ii. Give a correct proof of Punch’s First Theorem.

(b) **Punch’s Second Theorem:** For any finite sets $A, B$, if $|A|$ and $|B|$ are even then $|A \setminus B|$ is even.

**Punch’s Proof:** “The difference of two even numbers is an even number. Done.”

i. Now, Judy remarks that this other theorem is not even true. Give a counterexample that supports Judy’s contention.

ii. Judy also remarks that Punch’s “proof” relies on a false statement about set cardinalities. (Since the theorem is not true, there had to be a bug in the proof!) What is that false statement?

12. How many sequences of bits (0’s, 1’s) are there that each sequence has all of the following properties:

- Their length is either 3 or 5 or 7.
- Their middle bit is a 1.
- The number of 0’s they have equals the number of 1’s they have minus one.

13. A cookie shop has $k$ different flavors of cookies. Alex wishes to purchase cookies for his recitation, and he has enough money to buy up to 250 cookies. Assuming that he does not have to spend all of the money that he has, in how many ways can he purchase cookies? (For full credit, your solution should be in closed form, so summations with variable bounds!)
14. Give a combinatorial proof for the following identity:

\[ \sum_{k=0}^{r} \binom{m}{k} \binom{n}{r-k} = \binom{m+n}{r} \]

15. Recall (and remember!) that the sum of the squares of the first \( n \) positive integers is given by the following formula: 

\[ 1^2 + 2^2 + \cdots + (n-1)^2 + n^2 = n(n+1)(2n+1)/6. \]

(a) Using only the formula above, (no credit in part (a) for proof by induction, see part (b)), derive the following formula for the sum of the squares of first \( m \) odd positive integers. Show your work.

\[ 1^2 + 3^2 + 5^2 + \cdots + (2m-3)^2 + (2m-1)^2 = \frac{m(4m^2-1)}{3} \]

(b) Now prove by induction the formula from part (a).

16. Bob is recycling a set \( B \) of \( m \geq 1 \) distinguishable (he likes variety) bottles \( B = \{b_1, \ldots, b_m\} \) in a facility that has a set \( D \) of \( n \geq 2 \) distinguishable drums, \( D = \{d_1, \ldots, d_n\} \). When Bob shows up all the drums are empty. Each drum is large enough to hold by itself all of Bob’s \( m \) bottles. We call a deposit a way of placing the bottles in the drums, i.e., a function \( t : B \to D \). Each deposit may leave some drums (maybe none) empty. Let empty(\( t \)) be the set consisting of all the drums that are left empty by deposit \( t \). (Note that it might be the case that empty(\( t \)) = \( \emptyset \), depending on \( m, n \) and \( t \).)

Assume \( m \geq n \) and prove that there exist two different deposits, \( t_1 \) and \( t_2 \) such that empty(\( t_1 \)) = empty(\( t_2 \)).

17. Let \( A, B \) be two sets such that \( |A \cup B| = 12 \) and \( |A \cap B| = 8 \). Prove that \( 96 \leq |A \times B| \leq 100 \).

18. Consider the recurrence relation

\[ a_0 = 0 \quad a_1 = 1 \quad a_n = 2a_{n-1} - a_{n-2} + 1 \quad (\text{for } n \geq 2) \]

Express \( a_n \) as a polynomial in \( n \). (Hint: use the telescopic trick twice.) Then prove by induction the result you obtained.

19. Consider 33 distinct boolean expressions in exactly two variables. Prove that 3 or more of them must be logically equivalent.

20. In an All-Milky Way course the students receive their graded homeworks consisting of \( n \geq 2 \) problems, where each problem is given a score between 0 and \( m \geq 1 \). Assume that there are enough students (hence the galaxy-wide offering :) such that each set of possible scores on each problem is represented in the scores received by the students. For any two students \( a \) and \( b \) define Same(\( a, b \)) to be the set of all homework problems on which \( a \) and \( b \) got the same scores.

Use the Pigeonhole Principle to prove that there exist four students, \( a, b, c, d \) such that

- Same(\( a, b \)) = Same(\( c, d \)), and
- \( a \neq b \), and
- $c \neq d$, and
- $a \neq c$ OR $b \neq d$

21. For each statement below, decide whether it is TRUE or FALSE. In each case attach a very brief explanation of your answer.

(a) Let $(\Omega, \Pr)$ be a probability space with three outcomes. Let $E, F$ be two nonempty events in this space such that $\Pr[E \cup F] = \Pr[E] + \Pr[F]$. Then $E \cap F = \emptyset$.

(b) Let $A, B, C$ be three events of non-zero probability in a probability space $(\Omega, P)$. If $A \cap B = B \cap C$, $A \perp B$, and $B \perp C$ then $\Pr[A] = \Pr[C]$.

(c) If a probability space has an event of probability $2/3$ then it must have some outcome of probability at most $1/3$, true or false?

(d) Let $E, F$ be two events in a finite probability space. If $|E| = |F|$ then $\Pr[E] = \Pr[F]$, true or false?

(e) If $E, F$ are two events in a finite probability space such that $\Pr[E \cap F] > 0$ then $E$ and $F$ can be disjoint, true or false?

(f) Let $A, B$ be events in a finite probability space such that $\Pr[A] = 1/4$ and $\Pr[A \cup B] = 1/2$. Then, $1/4 \leq \Pr[B] \leq 1/2$, true or false?

(g) For any three events $E, F, G$ in the same probability space, if $E \perp F$ and $F \perp G$ then $E \perp G$.

22. Let $A, B, C$ be three events in the same probability space such that $A \subseteq B$, $A \subseteq C$, $B \perp C$, and $\Pr[A] = 1$. Prove that $\Pr[A \cap B \cap C] = \Pr[A] \Pr[B] \Pr[C]$.

23. Let $E, F$ be two events in a finite probability space such that $\Pr[E \cap F] > 0$. Prove that $\Pr[E \setminus F] + \Pr[F \setminus E] < \Pr[E \cup F]$. 