On Monday, June 17, we will have our first exam from 10AM to 12PM. The exam will take place in DRL A8, and last for 120 minutes. Please be in DRL A8 a few minutes early so we have time to seat everybody properly.

This is an exam review document with readings, a mock (practice) exam, and more practice problems. You should solve the practice exam while timing yourself.

Solutions to the practice exam will be posted Saturday June 15, in the afternoon.

More information about review sessions on Sunday will follow.

1 Readings

STUDY IN-DEPTH... ...the posted notes for lectures 1-8.

STUDY IN-DEPTH... ...the posted guides for recitations 1-3. (The recitation guide for Week 3 will be posted after recitation on Thursday.)

STUDY IN-DEPTH... ...the solutions to homework 1-3. Compare with your own solutions. (Homework 3 solutions will be available on Saturday and Sunday at office hours.)

STUDY IN-DEPTH... ...the solutions to the mock exam and the additional problems contained in this document, to be posted Saturday June 15, in the afternoon. Until then, try very hard to solve these on your own.

2 Mock Exam (120 minutes for 240 points)

1. (50 pts)

For each statement below, decide whether it is TRUE or FALSE and circle the right one. In each case attach a very brief explanation of your answer.

(a) \binom{100}{51} is strictly bigger than \binom{100}{49}.

(b) In Pascal’s Triangle, there exist four binomial coefficients c_1, c_2, c_3, c_4 such that c_1 = c_2 + c_3 + c_4.

(c) The boolean expressions \neg[(p \Rightarrow q) \vee q] and p \land \neg q are logically equivalent.

(d) Let A be a finite set. All functions f : A \rightarrow A are bijections.

(e) Let A, B be finite set. Then, there are exactly as many subsets of A \times B as there are functions with domain A and codomain 2^B.

(f) Assume that B is a set with 7 elements and that A is a set with 15 elements. Then, for any function f : A \rightarrow B there exist at least 3 distinct elements of A that are mapped by f to the same element of B.
(g) For any $A, B, C$ non-empty finite sets, let $m = |A| + |B| + |C| - |A \cup B \cup C|$ and $n = |A \cap B| + |B \cap C| + |C \cap A|$. Then $m > n$.

(h) If the set $A$ has $n$ elements then there are $n!$ injective functions with domain $A$ and codomain $A$.

(i) The Fibonacci number $F_{100}$ is even.

**Answer**

(a) **FALSE** Recall the following identity from lecture:

$$\binom{n}{k} = \binom{n}{n-k}$$

Then, the two sides of the equation are strictly equal.

(b) **TRUE** Consider the following values for $c_1, c_2, c_3,$ and $c_4$:

$$c_1 = \binom{3}{1} \quad c_2 = \binom{160}{0} \quad c_3 = \binom{160}{160} \quad c_4 = \binom{0}{0}$$

We thus see that:

$$c_1 = c_2 + c_3 + c_4$$

$$\binom{3}{1} = \binom{160}{0} + \binom{160}{160} + \binom{0}{0}$$

$$3 = 1 + 1 + 1$$

which we know to be true.

**Alternate Solution:**

We also could have solved this problem by applying Pascal’s Identity twice. Consider the following:

$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$$

$$= \binom{n-1}{k-1} + \binom{n-2}{k-1} + \binom{n-2}{k}$$

By choosing appropriate values for $n$ and $k$ (note that we require $n \geq 2, k \geq 1$, since combinations are undefined for negative values of $n$ or $k$), we can generate binomial coefficients for which the above equality holds. To illustrate, take $n = 5$ and $k = 3$. This gives us:

$$\binom{5}{3} = \binom{4}{2} + \binom{3}{2} + \binom{3}{3}$$

$$10 = 6 + 3 + 1$$

which we know to be true.

(c) **TRUE** We demonstrate this by showing that they have equivalent truth tables, as follows:
As proven in lecture, the number of injective functions from a domain of size \( T \) to a codomain of size \( F \) is given by:

\[
\binom{m}{T} = \frac{m!}{(m-T)!}.
\]

For a counterexample, consider the set \( A = \{1, 2\} \) and the function \( f : A \to A \) given by \( f(1) = f(2) = 1 \). The function is not injective, since \( f(1) = 1 = f(2) \). Therefore, it cannot be a bijection. Alternatively, we can argue that \( f \) is not surjective because there is no \( a \in A \) with \( f(a) = 2 \), and so it certainly also must not be bijective.

Let \( T \) be a bijection. Alternatively, we can argue that \( f \) is not injective, since \( \lceil \frac{m}{r} \rceil = f(1) = 1 = f(2) \). The function is not injective, since \( \lceil \frac{m}{r} \rceil \) is odd. Therefore, it cannot be surjective because there is no \( a \in A \) with \( f(a) = 2 \), and so it certainly also must not be bijective.

Let \( |A| = n \) and \( |B| = m \). From class, we know that:

\[
|A \times B| = |A| \cdot |B| = nm
\]

Furthermore, for a given set \( S \), we also know that \( |2^S| = 2^{|S|} \). Then, we conclude that the number of subsets of \( A \times B \) is given by:

\[
|2^{A \times B}| = 2^{|A \times B|} = 2^{nm}
\]

From class, we also know that the number of functions with domain \( X \) and codomain \( Y \) is given by \( |Y|^{|X|} \). In this problem, we have that \( X = A \) and \( Y = 2^B \). Then, the number of such functions is:

\[
|2^B|^{|A|} = (2^m)^n = 2^{nm}
\]

Therefore, we have shown that there are exactly as many subsets of \( A \times B \) as there are functions with domain \( A \) and codomain \( 2^B \).

Think of elements in \( A \) as pigeons and elements in \( B \) as holes; by GPHP, there is at least one element in \( B \) that are mapped with at least \( \lceil \frac{15}{7} \rceil = 3 \) elements in \( A \).

Rearranging the terms in PIE we have

\[
|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |B \cap C| - |C \cap A| + |A \cap B \cap C|
\]

\[
|A \cap B| + |B \cap C| + |C \cap A| = |A| + |B| + |C| - |A \cap B \cap C| - |A \cup B \cup C|
\]

\[
n = m + |A \cap B \cap C|
\]

Since \( |A \cap B \cap C| \geq 0 \), we can get \( n \geq m \). Thus, \( m > n \) is false.

**Alternative solution.** Disprove by an counterexample: When \( A, B, C \) are pairwise disjoint, \( |A \cup B \cup C| = |A| + |B| + |C| \), so \( m = 0 \); \( |A \cap B| = |B \cap C| = |C \cap A| = 0 \), so \( n = 0 \). Therefore, \( m = n = 0 \), and \( 0 > 0 \) is false.

As proven in lecture, the number of injective functions from a domain of size \( r \) to a codomain of size \( m \) is the same as the number of partial permutations of \( r \) out of \( m \), that is, \( \frac{m!}{(m-r)!} \). Here, both \( m \) and \( r \) equal \( n \) so the number of injective functions is \( \frac{n!}{(n-n)!} = \frac{n!}{0!} = \frac{n!}{1} = n! \).

Let’s see. \( F_0 = 0 \) is even, \( F_1 = 1 \) and \( F_2 = 1 \) are odd. \( F_3 = 2 \) is even again, and the pattern repeats. More generally, for any positive integer \( n \), \( F_{3n-2} \) and \( F_{3n-1} \) are odd and \( F_{3n} \) is even.

[This can be easily justified by induction on \( n \) but we are asking here only for a brief justification.]

Since \( 100 = 3 \cdot 34 - 2 \) it follows that \( F_{100} \) is odd.
2. (20pts)

Count the number of distinct sequences of bits (0’s, 1s) of length 101 such that:

- there are 3 more 1’s than 0’s in the sequence; and
- . . . also the middle bit is a 1.

**Answer**

Given the first constraint, we claim there must be exactly 52 1’s and 49 0’s in the sequence. Let $x$ be the total number of 0’s in the sequence. Then the total number of 1’s is $x + 3$, so we get

$$x + (x + 3) = 101$$
$$2x = 98$$
$$x = 49.$$

To count the total number of sequences, we use the Multiplication Rule:

*Step 1:* Place a 1 bit in the middle position.
*Step 2:* Place the remaining 1’s.
*Step 3:* Place the remaining 0’s.

Note first that there’s only one way to place a 1 bit in the middle position. For Step 2, we are left with 51 1’s to distribute among 101 − 1 = 100 remaining spots (since the middle spot is now occupied). This can be done in \(\binom{100}{51}\) ways. Finally, note that once the 1’s are in their positions, there’s only one way to place the 49 0’s in the 49 remaining positions. Specifically, the 0’s must go into the empty positions. Then, by the Multiplication Rule, our final answer is:

$$1 \cdot \binom{100}{51} \cdot 1 = \left(\frac{100!}{51!49!}\right)$$

3. (20pts)

Prove that for any $x, y, z \in \mathbb{Z}$ such that $x + 2y = z$, if $z - x$ is not divisible by 4 then $x + y + z$ is odd.

**Answer**

Proof by contradiction. Let $x, y, z \in \mathbb{Z}$ such that $x + 2y = z$ and $z - x$ is not divisible by 4. Assume toward a contradiction that $x + y + z$ is not odd.

Hence $x + y + z$ is even, and therefore $x + y + z = 2k$, for some $k \in \mathbb{Z}$. We also have that $z = x + 2y$, so we may write:

$$x + y + (x + 2y) = 2k$$
$$2x + 3y = 2k$$
$$3y = 2(k - x)$$

Therefore $3y$ must be even. It follows that $y$ must also be even. Indeed, if $y$ were odd then $3y$ would also be odd. So $y = 2\ell$ for some $\ell \in \mathbb{Z}.$
But then we have

\[ z - x = (x + 2y) - x = 2y = 4\ell \]

Since \( \ell \) is an integer, this means that \( 4 \mid z - x \), and we have reached a contradiction.

**Alternate Solution:** (direct proof).

We rewrite the given equation as

\[ 2y = z - x \]

Since \( y \in \mathbb{Z} \), this tells us that \( z - x \) is even. Additionally, since \( z - x \) is not divisible by 4, we know that \( \frac{z - x}{2} \) must be odd – if it were even, \( z - x \) would be divisible by 4.

We write

\[ z - x = 2(2k + 1), \quad k \in \mathbb{Z} \]

But then \( 2y = 2(2k + 1) \), so \( y = 2k + 1 \) is odd.

Finally, we know that \( 2x \) is even, since \( x \in \mathbb{Z} \). Putting this together gives us:

\[ x + y + z = (z - x) + 2x + y = 2(2k + 1) + 2x + 2k + 1 = 2(3k + x + 1) + 1 \]

As the sum of products of integers, \( 3k + x + 1 \) must be an integer, so \( x + y + z \) is odd.

4. (20pts)

Let \( A, B \) be any sets such that \( A \cap \{1, 2\} = B \cap \{1, 2\} \). Prove that the sets \( (A \setminus B) \cup (B \setminus A) \) and \( \{1, 2\} \) are disjoint.

**Answer**

If we can show that no element in \( ((A \setminus B) \cup (B \setminus A)) \) can be in \( \{1, 2\} \), then clearly the intersection of these 2 sets is empty and we are done.

We consider an arbitrary element \( x \in ((A \setminus B) \cup (B \setminus A)) \) and show that \( x \notin \{1, 2\} \). There are 2 cases:

Case 1: \( x \in A \setminus B \). Then \( x \in A \land x \notin B \). Assume for the sake of contradiction that \( x \in \{1, 2\} \). Then \( x \in A \cap \{1, 2\} \) as it is in both sets and \( x \notin B \cap \{1, 2\} \) as it is not in \( B \). Therefore, \( A \cap \{1, 2\} \neq B \cap \{1, 2\} \), a contradiction. So, \( x \notin \{1, 2\} \).

Case 2: This analogous to Case 1, by symmetry. (Switch \( A \) and \( B \) in all of the above statements).

In both cases \( x \notin \{1, 2\} \).

Since no element \( x \in ((A \setminus B) \cup (B \setminus A)) \) can be in \( \{1, 2\} \), these 2 sets are disjoint.

**Alternate Solution:**
We prove the contrapositive. Assume that the sets \((A \setminus B) \cup (B \setminus A)\) and \(\{1, 2\}\) are not disjoint, i.e.,
there exists some \(x \in ((A \setminus B) \cup (B \setminus A)) \cap \{1, 2\}\)
Therefore
\[
[(x \in A \setminus B) \lor (x \in B \setminus A)] \land (x = 1) \lor (x = 2)
\]
We have 4 cases.
Case 1: \((x \in A \setminus B) \land (x = 1)\) i.e., \(1 \in A \setminus B\). Then \(1 \in A\) and \(1 \notin B\). Since \(1 \in A\) we have \(1 \in A \cap \{1, 2\}\). Since \(1 \notin B\) we have \(1 \notin B \cap \{1, 2\}\). Therefore, \(A \cap \{1, 2\} \neq B \cap \{1, 2\}\).

The other three cases:
Case 2: \((x \in A \setminus B) \land (x = 2)\).
Case 3: \((x \in B \setminus A) \land (x = 1)\).
Case 4: \((x \in B \setminus A) \land (x = 2)\).
are analogous by symmetry. (Switch \(A\) and \(B\) and/or 1 and 2.)
In all four cases we have shown \(A \cap \{1, 2\} \neq B \cap \{1, 2\}\) and this proves the contrapositive.

5. (20pts)
Let \(n \in \mathbb{N}\) and \(n \geq 3\). Give a combinatorial proof (no other kinds of proofs will be accepted) for the following identity
\[
\binom{n+2}{3} = \binom{n}{1} + 2 \binom{n}{2} + \binom{n}{3}
\]
**Answer**

We ask the following question:

Since the weather is starting to get warmer Alex, Belinda, and Claudia decide to go get some ice cream. The ice cream shop, Clayton’s Creamery, has just 2 kinds of soft serve ice cream, and \(n\) kinds of hard ice cream. In how many ways can Alex, Belinda, and Claudia order 3 different kinds of ice cream if among the 3 of them it does not matter who gets which kind?

We can see that the LHS counts this question. There are \(n+2\) different kinds of ice cream, since there are 2 kinds of soft serve and \(n\) kinds of hard ice cream. Since we want to choose 3 of these kinds in any order, there are \(\binom{n+2}{3}\) ways to do this.

We now count the RHS. We can do this by casing on the number of kinds of soft ice cream that we choose. Since there are only 2 kinds of soft serve ice cream, we can choose 0, 1, or 2 kinds of soft ice cream.

**Case 1:** We choose 0 kinds of soft ice cream.
In this case we are choosing 3 kinds of hard ice cream. This can be done in \(\binom{n}{3}\) ways since the flavors of hard ice cream are distinguishable.

**Case 2:** We choose 1 kind of soft ice cream.
In this case we are choosing 1 kind of soft serve and 2 kinds of hard ice cream. This can be done in
2(\binom{n}{2}) ways. We can choose 2 kinds of hard ice cream in \binom{n}{2} ways, and then there are 2 ways to choose a kind of soft serve. Combining these, we have 2\binom{n}{2} ways.

**Case 3:** We choose 2 kinds of soft serve ice cream.
In this case we are choosing 2 kinds of soft serve and 1 kind of hard ice cream. There is only 1 way to choose 2 kinds of soft serve (since there are only 2 kinds). There are \binom{n}{1} ways to choose a kind of hard ice cream. Combining these, we have \binom{n}{1} ways.

Combining these 3 cases together by the Sum Rule, we see that the number of ways to select 3 kinds of ice cream is \binom{n}{1} + 2\binom{n}{2} + \binom{n}{3} which is exactly the RHS and we are done.

6. (20pts) My 6th grade teacher of Russian was unable to pay attention to what we were answering and it appeared to us that he was assigning grades completely randomly. Let’s assume that his grading rubric consisted of tossing a fair coin six times, counting the number \(k\) of heads and assigning the grade \(4 + k\) (our grades were in the 1-10 range).

(a) What was the probability that I would get a 10?
(b) What was the probability of the following event: “my grade was divisible by 4 or (non-exclusive or!) it was bigger than or equal to Lady Gaga’s shoe size (a 6)”?

**Answer**
We work with a uniform probability space \(\Omega\) with \(2^6\) outcomes. Each outcome is a sequence of length 6 of \(H\)’s and \(T\)’s and each outcome has probability \(1/2^6\).

(a) To get a 10 we must have \(k = 6\) therefore the event \(E\) of interest consists of the one outcome with exactly 6 heads (\(HHHHHH\)). Because \(\Omega\) is uniform, we can apply the following formula.

\[
\Pr[E] = \frac{|E|}{|\Omega|} = \frac{1}{2^6}
\]

(b) The grade can be (4 or 8) or (6 or 7 or 8 or 9 or 10) therefore \(k = 0\) or \(k \geq 2\). The event \(G\) of interest consists of sequences with no heads or with two or more heads. Its complement, \(\overline{G}\) consists of sequences with exactly one head. The one head can be in any of the 6 flips so there are 6 such sequences. Therefore

\[
\Pr[G] = 1 - \Pr[\overline{G}] = 1 - \frac{|\overline{G}|}{|\Omega|} = 1 - \frac{6}{2^6}
\]

7. (25pts)
Let

\[
R_n = \sum_{k=1}^{2n} (-1)^{k+1}k \quad \text{for } n \geq 1
\]

(a) Compute \(R_1, R_2, R_3\). Guess a simple way to express \(R_n\) in terms of \(n\). Prove your guess by induction.

(b) Prove by induction that for all \(n \geq 1\) we have

\[1 + 3 + 5 + \cdots + (2n - 1) = n^2\]
(c) Use the identity in part (b) and other identities that you were supposed to memorize to prove the identity in part (a).

**Answer**

(a) \( R_1 = 1 - 2 = -1. \)
\( R_2 = 1 - 2 + 3 - 4 = -2. \)
\( R_3 = 1 - 2 + 3 - 4 + 5 - 6 = -3. \)

We guess \( \forall n \geq 1 \ R_n = -n. \) And we prove it by induction. We already have the base case \( R_1 = -1. \)

**Induction Step:** Let \( k \geq 1 \) be an arbitrary integer. Assume \( R_k = -k \) (IH). We want to show \( R_{k+1} = -(k + 1) \)

\[
R_{k+1} = \sum_{i=1}^{2(k+1)} (-1)^{i+1} i \quad \text{(by the given def of } R_n) \]
\[
= \sum_{i=1}^{2k+2} (-1)^{i+1} i \]
\[
= \sum_{i=1}^{2k} (-1)^{i+1} i + (2k + 1) - (2k + 2) \]
\[
= R_k + (2k + 1) - (2k + 2) \quad \text{(by the given def of } R_n) \]
\[
= -k + (2k + 1) - (2k + 2) \quad \text{(by IH)} \]
\[
= -k + 2k + 1 - 2k - 2 \]
\[
= -k - 1 \]
\[
= -(k + 1) \]

Done.

(b) **Base Case:** \( n = 1. \) \( 1 = 1^2 \) Check.

**Induction Step:** Let \( k \geq 1 \) arbitrary, fixed. Assume \( 1 + 3 + \cdots + (2k - 1) = k^2 \) (IH). We want to show that \( 1 + 3 + \cdots + (2(k + 1) - 1) = (k + 1)^2 \)

\[
1 + 3 + \cdots + (2(k + 1) - 1) = 1 + 3 + \cdots + (2k + 1) \]
\[
= (1 + 3 + \cdots + 2k - 1) + (2k + 1) \]
\[
= k^2 + (2k + 1) \quad \text{(by IH)} \]
\[
= (k + 1)^2 \]

Done.
(c) We know that
\[ 1 + 2 + 3 + 4 + \cdots + (2n) = \frac{2n(2n+1)}{2} \]
Let’s write the definition of \( R_n \) in part (a) like this
\[ 1 - 2 + 3 - 4 + \cdots - (2n) = R_n \]
Adding LHS and the RHS of these two equalities and canceling \( 2 - 2 = 0 \) etc., we have
\[ 1 + 1 + 3 + 3 + \cdots + (2n - 1) + (2n - 1) = \frac{2n(2n+1)}{2} + R_n \]
By part (b)
\[ 2 \cdot n^2 = \frac{2n(2n+1)}{2} + R_n \]
Now it’s just algebra
\[ R_n = 2n^2 - \frac{2n(2n+1)}{2} = 2n^2 - 2n^2 - n = -n \]
8. (20pts)
Alice has a strange coin that shows the number 3 on one side and the number 5 on the other. Still, the coin is fair. Bob has strange die that shows the numbers 5, 6, 7, 8, 9, 10 on its six faces. Still, the die is fair. Alice flips the coin and, independently, Bob rolls the die. What is the probability that the number on the die is divisible by the number on the coin?

**Answer**
Let \( c \) be the number shown by the coin and \( d \) be the number shown by the die. Let \( E \) denote the event that \( c \mid d \).

**Solution 1** The sample space, \( \Omega \), consists of all possible ordered pairs \((c,d)\), where \( c \) denotes the result of the coin flip and \( d \) denotes the result of the die roll. Because the coin and die are fair, and the flip and roll are independent, we have a uniform probability space with \( |\Omega| = 2 \times 6 = 12 \) outcomes. \( E \) consists of 4 of these outcomes:
\[ E = \{(c = 3, d = 6), (c = 3, d = 9), (c = 5, d = 5), (c = 5, d = 10)\}. \]
Therefore, we see that
\[ \Pr[E] = \frac{|E|}{|\Omega|} = \frac{4}{12} = \frac{1}{3}. \]

**Solution 2** (This solution gets the right result but relies, as is commonly the case, on more complex assumptions.) Since the die is fair, we have \( \Pr[d = 6 \text{ or } d = 9] = (1/6) + (1/6) = 2/6 = 1/3 \). Similarly, \( \Pr[d = 5 \text{ or } d = 10] = 1/3 \). Now, since the coin flip and the die roll are independent, and since the coin is fair
\[ \Pr[(d = 6 \text{ or } d = 9) \text{ and } c = 3] = \Pr[d = 6 \text{ or } d = 9] \Pr[c = 3] = (1/3)(1/2) = 1/6. \]
Similarly,
\[ \Pr[(d = 5 \text{ or } d = 10) \text{ and } c = 5] = \Pr[d = 5 \text{ or } d = 10] \Pr[c = 5] = (1/3)(1/2) = 1/6. \]
These events are disjoint, so the probability we seek is \( (1/6) + (1/6) = 1/3. \)
9. (15pts)

A lottery urn contains \( n \geq 2 \) distinct balls labeled with the numbers 1, \ldots, \( n \). You extract two distinct balls from the urn. Suppose they are labeled \( i \) and \( j \). You compute \( i + j \) and write down the answer on a piece of paper. Then you put the two balls back.

You repeat this \( m \) times. What is the smallest value of \( m \) that ensures (no probabilities in this problem!) that you will end up writing the same number at least twice on the piece of paper. Prove your answer.

**Answer**

The smallest number you can write down is \( 1 + 2 = 3 \). The largest number you can write down is \((n - 1) + n = 2n - 1\).

In fact, every \( 3 \leq k \leq 2n - 1 \) could be one of the numbers written down. Indeed, when \( 3 \leq k \leq n + 1 \) then \( k = 1 + (k - 1) \) because \( 2 \leq k - 1 \leq n \). And when \( n + 2 \leq k \leq 2n - 1 \) then \( k = n + (k - n) \) because \( 2 \leq k - n \leq n - 1 \).

The piece of paper can contain any of the numbers in \([3, (2n - 1)]\) hence \( 2n - 1 - 3 + 1 = 2n - 3 \) distinct numbers (pigeonholes). By PHP, if \( m = (2n - 3) + 1 = 2n - 2 \) then at least one number will be written down at least twice.

However, PHP does not tell us specifically whether \( 2n - 2 \) is the smallest such \( m \). But if \( m \) is smaller, namely \( m = 2n - 3 \), then we get all distinct numbers with the following \( 2n - 3 \) extractions: \( \{1, i\} \) for \( i = 2, \ldots, n \) (\( n - 1 \) extractions) followed by \( \{j, n\} \) for \( j = 2, \ldots, n - 1 \) (\( n - 2 \) extractions). If \( m \) is even smaller we just consider a subset of these extractions.

10. (10pts)

Let \( X \) be a nonempty finite set. Consider the set \( \mathcal{W} = \{(A, B) \mid A, B \in 2^X \land A \subseteq B\} \).

Prove that \( \mathcal{W} \) has exactly as many elements as there are functions with domain \( X \) and codomain \( \{1, 2, 3\} \).

**Answer**

We find the sizes of these two sets: First, we consider \( \mathcal{W} \). We are counting ordered pairs \( (A, B) \) such that \( A \subseteq B \subseteq X \). For this constraint to hold, each element \( x \in X \) must have either \( x \not\in B \) (and therefore \( x \not\in A \)), \( x \in B \setminus A \) or \( x \in A \) (and therefore \( x \in B \)). (You may recall this from an earlier homework.) We can construct an ordered pair \( (A, B) \) in \( |X| \) steps; in each step, one element \( x \in X \) is assigned one of the 3 states \( x \not\in B \), \( x \in B \setminus A \) or \( x \in A \). Thus, this whole process can be done in \( 3^{|X|} \) ways.

For the size of the other set we note that the size of the codomain is 3 therefore (as shown in lecture) there are \( 3^{|X|} \) functions with domain \( X \) and codomain \( \{1, 2, 3\} \). Thus, \( \mathcal{W} \) and the set of functions \( f \) described in the problem have equal size.

**NOTE** This also follows from the Bijection Rule. Indeed, in each \( (A, B) \) of \( \mathcal{W} \), each \( x \in X \) must be mapped to one of 3 states "not in \( B \), "in \( B \setminus A \)" and "in \( A \). This establishes a one-to-one correspondence between the pairs \( (A, B) \) in \( \mathcal{W} \) and the functions with domain \( X \) and a codomain formed by these three states. As this codomain is of the same size as the set \( \{1, 2, 3\} \), then there are also an equal number of functions with domain \( X \) and codomain \( \{1, 2, 3\} \).
3 Additional Problems

1. For each statement below, decide whether it is TRUE or FALSE and circle the right one. In each case attach a very brief explanation of your answer.

(a) There are exactly three surjective functions with domain \( \{1, 2\} \) and codomain \( \{a, b\} \).
(b) Exactly two of the following three boolean expressions: \( p \Rightarrow q \), \( p \land \neg q \), and \( \neg p \lor q \) are logically equivalent.
(c) Let \( A \) be a finite set. For any function \( f : A \rightarrow A \) we have \( |\text{Ran}(f)| = |A| \).
(d) There exist two distinct functions with domain and codomain \( \{a, b\} \) that are their own inverses.
(e) For any two finite sets \( A, B \), \( |2^{A \times B}| > |2^A \times 2^B| \).
(f) Recall that for any \( n = 0, 1, 2, 3, \ldots \) row \( n \) of the Pascal Triangle contains the binomial coefficients of the form \( \binom{n}{k} \) for \( k = 0, 1, \ldots, n \). \( \binom{7}{4} \) can be expressed as a sum of binomial coefficients from row 5.

ANSWER

(a) FALSE.

There are exactly two. One that maps 1 to \( a \) and 2 to \( b \) and one that maps 1 to \( b \) and 2 to \( a \).

(b) TRUE.

The first and the third are logically equivalent. The second is logically equivalent to the negation of the first. We see this with a quick table:

<table>
<thead>
<tr>
<th></th>
<th></th>
<th>\neg p</th>
<th>\neg q</th>
<th>p \Rightarrow q</th>
<th>p \land \neg q</th>
<th>\neg p \lor q</th>
</tr>
</thead>
<tbody>
<tr>
<td>T</td>
<td>T</td>
<td>F</td>
<td>F</td>
<td>T</td>
<td>F</td>
<td>T</td>
</tr>
<tr>
<td>T</td>
<td>F</td>
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<tr>
<td>F</td>
<td>F</td>
<td>T</td>
<td>T</td>
<td>F</td>
<td>F</td>
<td>T</td>
</tr>
</tbody>
</table>

(c) FALSE.

Counterexample: Let \( A = \{0, 1\} \), and \( f(0) = f(1) = 0 \).

Then \( |A| = 2 \) but \( |\text{Ran}(f)| = 1 \) since \( \text{Ran}(f) = \{0\} \).

(d) TRUE.

Consider the following two distinct functions \( f : A \rightarrow A \) and \( g : A \rightarrow A \):

\[
\begin{align*}
f(a) &= a & g(a) &= b \\
f(b) &= b & g(b) &= a
\end{align*}
\]

Observe that we have the following:

\[
\begin{align*}
f(f(a)) &= f(a) = a & g(g(a)) &= g(b) = a \\
f(f(b)) &= f(b) = b & g(g(b)) &= g(a) = b
\end{align*}
\]

Since \( \forall x \in A, f(f(x)) = x \) and \( g(g(x)) = x \), \( f \) and \( g \) are each their own inverse. Thus, the statement is true.
(e) FALSE.
Take \( A = \{a\} \) and \( B = \{b\} \). Then \( A \times B = \{(a, b)\} \).
So \( |A| = |B| = |A \times B| = 1 \).
\[ |2^{A \times B}| = 2^{|A \times B|} = 2, \]
\[ |2^A \times 2^B| = |2^A| \times |2^B| = |2^A| \times 2^{|B|} = 2 \times 2 = 4. \]
\( 2 \not\geq 4 \).

(f) TRUE.
By Pascal’s Identity
\[ \binom{7}{4} = \binom{6}{3} + \binom{6}{4} = \binom{5}{2} + \binom{5}{3} + \binom{5}{3} + \binom{5}{4} \]

2. Give a boolean expression \( e \) with three variables \( p, q, r \) such that \( e \) has the following properties:
- \( e = T \) when \( p = q = T \) and \( r = F \), AND
- \( e = F \) when \( p = F \) and \( q = r = T \).

Also, construct a truth table for \( e \). Make sure to include all intermediate propositions as a separate column. (Yes, there are many possible answers.)

**Answer**
This is one such possible answer: Let \( e = p \land q \land \neg r \).

Here is a truth table for \( e = p \land q \land \neg r \):

<table>
<thead>
<tr>
<th>p</th>
<th>q</th>
<th>r</th>
<th>\neg r</th>
<th>p \land q</th>
<th>e = p \land q \land \neg r</th>
</tr>
</thead>
<tbody>
<tr>
<td>T</td>
<td>T</td>
<td>T</td>
<td>T</td>
<td>T</td>
<td>F</td>
</tr>
<tr>
<td>T</td>
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<td>F</td>
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<td>F</td>
<td>F</td>
<td>T</td>
<td>F</td>
<td>F</td>
</tr>
</tbody>
</table>

Note that \( e \) has the properties given by the problem. That is, \( e = T \) when \( p = q = T \) and \( r = F \), and \( e = F \) when \( p = F \) and \( q = r = T \).

You can also solve this problem systematically. Note that \( p \land q \land \neg r \) is \( T \) exactly when \( p = q = T \) and \( r = F \). Similarly, \( \neg p \land q \land r \) is \( T \) exactly when \( p = F \) and \( q = r = T \), therefore \( \neg(\neg p \land q \land r) \) is \( F \) exactly when \( p = F \) and \( q = r = T \). If there are more such constraints, you find more such conjunctions or negated conjunctions.

Then, we can also have \( e = (p \land q \land \neg r) \lor \neg(\neg p \land q \land r) \).

3. In how many different ways can we arrange *all* the letters from the English alphabet (26 characters) in a sequence such that:
• each letter occurs exactly once, AND
• the 5 vowels (a,e,i,o,u) occur in 5 consecutive positions.

**Answer**

We can solve this using the multiplication rule.

*Step 1:* Arrange the consonants. \((21! \text{ ways})\)

*Step 2:* Choose a position for the vowels. \((22 \text{ ways})\)

*Step 3:* Arrange the vowels. \((5! \text{ ways})\)

Thus, our final answer is \(21! \times 22 \times 5! = 22! \times 5!\)

**Alternate Solution:**

Since the vowels appear in consecutive positions, we can think of them as a block. There are 21 consonants plus the vowel block, giving us a total of 22 elements. Again, we’ll solve this using the multiplication rule.

*Step 1:* Arrange the 22 elements (21 consonants plus the 5 vowel block). \((22! \text{ ways})\)

*Step 2:* Arrange the vowels. \((5! \text{ ways})\)

**Answer:** \(22! \times 5!\)

4. Give combinatorial proofs (no other kind of proofs will be accepted) for the following identities:

(a) \(\binom{n}{r} \binom{r}{k} = \binom{n}{k} \binom{n-k}{r-k}\) (where \(k \leq r \leq n\))

(b) \(\sum_{i=k}^{n} \binom{i}{k} = \binom{n+1}{k+1}\) (where \(k \leq n\))

**Answer**

(a) Our question is as follows:

Given \(n\) pieces of (distinguishable) sushi, how many ways can we choose \(r\) pieces to eat if we also want to put wasabi on \(k\) of them?

**LHS:** We first choose the \(r\) pieces of sushi to eat \(\binom{n}{r}\), and then we choose \(k\) of them out of \(r\) to put wasabi on \(\binom{r}{k}\).

**RHS:** We first choose the \(k\) pieces of sushi to both eat and put wasabi on. This can be done in \(\binom{n}{k}\) ways. Then we choose the \(r-k\) ones without wasabi to eat from the remaining \(n-k\), which can be done in \(\binom{n-k}{r-k}\) ways.

(b) Our question is as follows:

AJ is writing a new song. He wants to include exactly \(k+1\) (distinguishable) chords at some point in his song out of a possible \(n+1\) chords (labeling them \(\{1, 2, \ldots, n+1\}\). How many collections of \(k+1\) can he make?
LHS: We split it up into cases. The first is that we include chord 1. There are \( \binom{n}{k} \) ways to make such a collection. The next is we don’t include chord 1 but include chord 2. There are \( \binom{n-1}{k} \) such collections. The next is we don’t include chords 1 and 2 but include chord 3. This can be done \( \binom{n-2}{k} \) ways. We can do this up until chord \( n-k \) because then we must include the next \( k \) chords. We can express this as the sum \( \sum_{i=k}^{n} \binom{n}{i} \)

RHS: We are asked to pick \( k+1 \) chords from a total of \( n+1 \) distinguishable chords. There are \( \binom{n+1}{k+1} \) ways to do this.

5. Prove that for any integers \( a, b \in \mathbb{Z} \) we have \( a^2 - 4b \neq 2 \).

Answer

Assume, toward a contradiction that \( a^2 - 4b = 2 \). Then \( a^2 = 4b + 2 = 2(2b + 1) \) and therefore \( a^2 \) is even. We have shown in lecture that this implies \( a \) is also even.

Let \( a = 2k \) for some integer \( k \).

\[
\begin{align*}
(2k)^2 &= 4b + 2 \\
4k^2 &= 4b + 2 \\
2k^2 &= 2b + 1 \\
2k^2 - 2b &= 1 \\
2(k^2 - b) &= 1
\end{align*}
\]

Now, because \( k^2 - b \) is an integer, it must be the case that \( 2(k^2 - b) \) is even. Therefore 1 is even. Contradiction.

6. Consider \( n \) (distinguishable) bins labeled \( B_1, \ldots, B_n \) and \( r \) indistinguishable (identical) marbles. We wish to put the \( r \) marbles into the \( n \) bins in such a way that each bin will contain at least one marble and at least three of the bins will contain two or more marbles. Assume \( r \geq n + 3 \). In how many different ways can this be done?

Answer

First we compute the number of ways to put \( r \) marbles into the \( n \) bins such that each bin will contain at least one marble. It’s

\[
\binom{n + (r - n) - 1}{r - n} = \binom{r - 1}{r - n}
\]

But not all of these ways ensure that at least three of the bins will contain two or more marbles. To correct for this over counting, we will count these bad cases and subtract from the number above.

If it is not the case that at least three of the bins will contain two or more marbles then only exactly zero or exactly one or exactly two bins contain two or more marbles. Note that these are disjoint cases, we do not have to worry about counting a bad case twice.

Case 1: Exactly zero bins with two or more marbles.

This is impossible. We have \( r \geq n + 3 \) marbles, and if we distribute one to each, we will be left with \( r - n \geq 3 \) marbles. As no more bins have more marbles, we have not distributed all the marbles.

Case 2: Exactly one bins with two or more marbles.
This means that all \( r - n \) additional marbles go into one bin. There are \( n \) ways to choose that one bin, so this can be done in \( n \) ways.

**Case 3: Exactly two bins with two or more marbles.**

This means means that all \( r - n \) additional marbles go into two bins. We can do this in three steps. In Step 1 we choose two of the bins, in \( \binom{n}{2} \) ways. In Step 2 we put one more marble in each of the two bins (recall that each bin already had a marble). This guarantees that both bins will have two or more marbles. This is done in one way and we are left with \( r - n - 2 \) marbles. In Step 3 we put the remaining \( r - n - 2 \) marbles in the two (distinguishable) bins we chose, any way we want. So this can be done in \( (r - n - 2) + 1 = r - n - 1 \) ways (think \( r - n - 2 \) stars and 1 bar). By multiplication rule the count in the “exactly two” case is:

\[
\binom{n}{2} (r - n - 1)
\]

Putting this all together, the total number of ways is:

\[
\binom{r - 1}{r - n} - n - \binom{n}{2} (r - n - 1)
\]

7. For each statement below, decide whether it is TRUE or FALSE. In each case attach a very brief explanation of your answer.

(a) The word QWERTY has 6! anagrams. (Recall that a word is a valid anagram of itself.)

(b) The contrapositive of \( p \rightarrow q \) is logically equivalent to \( p \land \neg q \).

(c) For any \( 2 \leq k < n \), if \( A \) has \( n \) elements then the number of subsets of \( A \) of \( k \) elements is \( \frac{n!}{(n-k)!} \).

(d) If the set \( A \) has \( n \) elements then there are \( n! \) injective functions with domain \( A \) and codomain \( A \).

(e) There is no set \( X \) such that \( 2^X = \emptyset \).

Answer

(a) TRUE. There are no multiple occurrences of the same letter so the bag is the same as the set and they have the same number of permutations. Alternatively, you could state

\[
\frac{6!}{1!1!1!1!1!} = 6!
\]

(b) FALSE. The contrapositive is \( \neg q \Rightarrow \neg p \) and it is not logically equivalent to \( p \land \neg q \) because, for example, for the truth assignment \( p = T, q = F \) we have \( \neg q \Rightarrow \neg p = F \) but \( p \land \neg q = T \). *Alternative explanation:* We know from class that \( p \rightarrow q \) is logically equivalent to its contrapositive. However we also know that \( p \land \neg q \) is logically equivalent to the negation of \( p \rightarrow q \). A proposition cannot be logically to its negation.

(c) FALSE. \( (n)_k \) is the number of permutations (not subsets) of \( k \) elements out of \( n \). *Alternative explanation:* The number of subsets of size \( k \) of a set of \( n \) elements is different, namely \( \binom{n}{k} \).
(d) TRUE. The injective functions from $A$ to $A$ correspond one-to-one to the permutations of the elements of $A$ and there are $n!$ such permutations.

(Not required but FYI: The one-to-one correspondence can be described as follows. Let $A = \{a_1, \ldots, a_n\}$ and let $f : A \to A$. When $f$ is injective, the sequence $f(a_1), \ldots, f(a_n)$ consists of $n$ distinct elements of $A$ and is therefore a permutation of $A$. Conversely, given a permutation $b_1, \ldots, b_n$ of the elements of $A$ then the function defined by $f(a_i) = b_i$, $i = 1, \ldots, n$ must be injective because the $b_i$’s are distinct.)

(e) TRUE. For any $X$, the set $2^X$ contains at least one element, namely $\emptyset$ (the empty subset of $X$).

8. The Taney Dragons are going to the Little League World Series! In appreciation, each of the 12 distinct team members (players) can pick 2 hats from a supply of red (Philly Phillies), blue (Boston Red Sox), and green (Ploiesti Frackers) hats. For each color, the supply is unlimited. For each of the three questions below (see also next page), give the answer and an explanation of how you derived it. No proofs required.

In how many different ways can the hat picking be done if:

(a) There is no ordering among the two hats that each player picks, and both hats can even be of the same color.

(b) The ordering matters and the two hats have a different color: let’s say each player picks a hat to wear in the morning and then a hat (of a different color) to wear in the afternoon.

(c) What is the count for part (8a) above, if you also know that at least one of the hats that Dragon’s pitcher Mo’ne Davis picks is red.

Answer

(a) Each player can pick an unordered set of two hats of different color, and there are 3 such options, or two hats of the same color, and there are 3 such options. By the sum (addition) rule each player has 6 options. Because the supplies are unlimited, a hat picking is just like sequence of length 12 of options. There are $6^{12}$ such sequences (equivalently $6^{12}$ functions from the set of 12 players to the set of 6 options). Hence $6^{12}$ ways.

(b) Each player has 3 options for her/his morning hat and once he/she picks that, is left with 2 options for for her/his afternoon hat. Therefore, by the multiplication rule, each player has $3 \cdot 2 = 6$ options. Again we count the number of sequences of options (or functions from the set of players to the set of options) and we get $6^{12}$ ways.

(c) We are back to the 6 options described in the solution to part (8a) above. But now these 6 options are only available to Mo’ne’s 11 teammates so they can pick hats in $6^{11}$ ways. Mo’ne herself must pick a red Philly hat and for her other hat she can pick any of the colors so she has 3 options. Thus she only has 3 options.

By the multiplication rule the total number of ways for this part is $3 \cdot 6^{11}$.

9. Recall from homework that the boolean expression $e_2$ is a logical consequence of the boolean expression $e_1$ if every truth assignment to the variables that makes $e_1$ true also makes $e_2$ true.

16
Let $x, y$ be arbitrary boolean variables. Prove, using truth tables, that $x \rightarrow y$ is a logical consequence of $\neg x \land y$.

**Answer**

We compute the truth table:

<table>
<thead>
<tr>
<th>$x$</th>
<th>$y$</th>
<th>$\neg x$</th>
<th>$\neg x \land y$</th>
<th>$x \rightarrow y$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$T$</td>
<td>$T$</td>
<td>$F$</td>
<td>$F$</td>
<td>$T$</td>
</tr>
<tr>
<td>$T$</td>
<td>$F$</td>
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</tr>
<tr>
<td>$F$</td>
<td>$F$</td>
<td>$T$</td>
<td>$F$</td>
<td>$T$</td>
</tr>
</tbody>
</table>

To prove if this is a logical consequence, we need check to that every assignment of $x, y$ that makes $\neg x \land y$ true also makes $x \rightarrow y$ true. Indeed, there is only one assignment $x = F, y = T$ that makes the first expression true, and the same assignment makes the second expression true. So, we have that $x \rightarrow y$ is a logical consequence of $\neg x \land y$.

10. Provide examples for the following. You do not have to prove that they work.

(a) For arbitrary $n \geq 1$, give an example of a set $Y$ and a function $f : [1..n] \rightarrow Y$ such that $f$ is injective but not surjective.

(b) For arbitrary $n \geq 2$, give an example of a set $X$ and a function $g : X \rightarrow [1..n]$ that is not injective and moreover $|\text{Ran}(g)| = n - 1$.

**Answer**

(a) Define $Y = [1..n+1]$ and for each $1 \leq k \leq n$ define $f(k) = k$.

(Not required but FYI: The function is clearly injective since every value 1 to $n$ maps to itself, and it is not surjective because there is no $k \in [1..n]$ such that $f(k) = n + 1$.)

(b) Define $X = [1..n]$ and for each $1 \leq k \leq n$ define

$$g(k) = \begin{cases} k & \text{if } 1 \leq k \leq n - 1 \\ n - 1 & \text{if } k = n \end{cases}$$

(Not required but FYI: The function is not injective because two different values map to $n - 1$, specifically $g(n - 1) = n - 1 = g(n)$. Moreover $\text{Ran}(g) = \{1, \ldots, n - 1\}$ therefore $|\text{Ran}(g)| = n - 1$.)

11. Punch happily tells Judy that he proved two new theorems and he shares his proofs with her.

(a) **Punch’s First Theorem**: If $n$ is odd then $n^2 - 1$ is a multiple of 4.

**Punch’s Proof**: “We prove the contrapositive instead. Suppose $n$ is even, then $n^2$ is even, then $n^2 - 1$ is odd so it cannot be a multiple of 4. Done.” Upon reading these, Judy remarks that while the theorem is true, the proof is not proving the theorem, but another statement, which is not the contrapositive of of the theorem.

i. What is the contrapositive of the theorem and what statement is Punch actually proving?

ii. Give a correct proof of Punch’s First Theorem.
(b) **Punch’s Second Theorem:** For any finite sets $A, B$, if $|A|$ and $|B|$ are even then $|A \setminus B|$ is even.

**Punch’s Proof:** “The difference of two even numbers is an even number. Done.”

i. Now, Judy remarks that this other theorem is not even true. Give a counterexample that supports Judy’s contention.

ii. Judy also remarks that Punch’s “proof” relies on a false statement about set cardinalities. (Since the theorem is not true, there had to be a bug in the proof!) What is that false statement?

**Answer**

(a) i. The contrapositive is:
   “If $n^2 - 1$ is not a multiple of 4 then $n$ is even (or you can say “is not odd”).”

What Punch actually proved is:
“If $n$ is even then $n^2 - 1$ is not a multiple of 4.”

(Not required but FYI: This is the converse of the contrapositive and in general it’s not logically equivalent to the theorem.)

ii. Direct proof: if $n$ is odd then $n = 2k + 1$ for some integer $k$. Thus,

$$n^2 - 1 = (2k + 1)^2 - 1 = 4k^2 + 4k + 1 - 1 = 4(k^2 + k)$$

which is clearly divisible by 4.

**Alternate Solution:** Suppose that we attempt to prove the contrapositive. This leads to the following strange but interesting proof:

Suppose $n^2 - 1$ is not a multiple of 4 and W.T.S. $n$ is even.

*Case 1:* $n^2 - 1$ is odd. In this case $n^2$ is even and therefore $n$ is even.

*Case 2:* $n^2 - 1$ is even but still not a multiple of 4. In this case $n^2$ is odd so $n = 2k + 1$ for some integer $k$ and then $n^2 - 1 = (2k + 1)^2 - 1 = 4k^2 + 4k + 1 - 1 = 4(k^2 + k)$ hence a multiple of 4. But our statement says that $n^2 - 1$ couldn’t have been a multiple of 4, and thus this case is impossible. Therefore since we showed in this case the premise is false, and false implies anything, the statement still holds true!

$n$ is even in both cases. Done.

The strange thing is that in the process of doing a proof by contrapositive you essentially discover the direct proof too, see inside Case 2.

(b) i. Since the claim concerns all finite sets, showing one counterexample is enough to disprove the claim. A counterexample is $A = \{1, 2\}, B = \{2, 3\}$ therefore $A \setminus B = \{1\}$. This gives us $|A \setminus B| = 1$, but $|A| - |B| = 0$.

ii. Punch assumes that $|A \setminus B| = |A| - |B|$. (Not Required but FYI: In general, $|A \setminus B| = |A| - |A \cap B|$.)

12. How many sequences of bits (0’s, 1’s) are there that each sequence has all of the following properties:

- Their length is either 3 or 5 or 7.
• Their middle bit is a 1.
• The number of 0’s they have equals the number of 1’s they have minus one.

**Answer**

We partition the possible sequences into three cases by their lengths.

**Case 1:** Sequences with 3 bits Since the middle bit must be a 1 and the bit must contain a total of two 1’s and one 0, there are only 2 ways to arrange the bits: 011 and 110.

**Case 2:** Sequences with 5 bits Since the middle bit must be a 1, we only need to consider the possible arrangements of the remaining 1’s and 0’s. To do this, we choose the 2 of the 4 non-middle positions to place the 1’s and then place the 0’s in the remaining spots. This can be done in $\binom{4}{2}$ ways.

**Case 3:** Sequences with 7 bits Again, the middle bit must be a 1, so we only need to consider the possible arrangements of the remaining 1’s and 0’s. To do this, we choose 3 of the 6 non-middle positions to place the 1’s and then place the 0’s in the remaining spots. This can be done in $\binom{6}{3}$ ways.

Since the cases are disjoint, we can apply the Sum Rule to get

$$2 + \binom{4}{2} + \binom{6}{3} = 2 + \frac{4!}{2!2!} + \frac{6!}{3!3!} = 28$$

13. A cookie shop has $k$ different flavors of cookies. Alex wishes to purchase cookies for his recitation, and he has enough money to buy up to 250 cookies. Assuming that he does not have to spend all of the money that he has, in how many ways can he purchase cookies? (For full credit, your solution should be in closed form, so no summations with variable bounds!)

**Answer**

The problem essentially asks for the number of ways to distribute 250 cookies into distinguishable categories.

Each of these cookies will be either one of the $k$ flavors or “not purchased,” so we can imagine $k + 1$ bins to place the cookies in. Since any two unpurchased cookies or cookies of the same flavor are indistinguishable, we can use the stars and bars method, where the cookies are the stars and the $(k + 1) - 1 = k$ dividers between the $k + 1$ categories are the bars.

For example, if Alex purchases 1 cookie of each of $n$ varieties and then 250 - $n$ potential cookies are left unpurchased (because he’s feeling stingy), then we would represent this with one star followed by one bar $k$ times, and then 250 - $k$ stars representing unpurchased cookies.

Thus, using the formula for the stars and bars method, the answer is $\binom{250+k}{k}$.

**Alternative (incomplete) solution:** You could choose to sum over all possibilities. For each $i$ between 0 and 250 equal to number of cookies purchased, we want to count the number of ways we can split $i$ up into the $k$ varieties. This can still be done with stars and bars, but we now have $k$ bins, one for each variety. Summing, we have

$$\sum_{i=0}^{250} \binom{i + k - 1}{k - 1} = \sum_{j=k-1}^{250+k-1} \binom{j}{k-1}$$
If you leave it like this, you do not get full credit. But this sum actually equals \( {250+k \choose k} \) (HW2 problem 7-d). In fact these two alternative solutions provide a combinatorial proof for (HW2 problem 7-d).

14. Give a combinatorial proof for the following identity:

\[
\sum_{k=0}^{r} \binom{m}{k} \binom{n}{r-k} = \binom{m+n}{r}
\]

**Answer**

We pose the following counting question.

Shawn bought \( m \) (distinguishable) math textbooks and \( n \) (distinguishable) CS textbooks as summer reading, but Seth forgot to buy books to read, so Shawn agrees to give him \( r \) of his textbooks. How many different combinations \( r \) textbooks consisting of Shawn’s math and CS textbooks can Shawn give Seth?

We need to solve this problem in 2 ways which yield the RHS and LHS respectively.

First, we look at the RHS. Since we are choosing \( r \) books from \( m+n \) total distinguishable books, we simply apply the definition of a combination to have \( \binom{m+n}{r} \).

We now look at the LHS. Since we see a summation, we consider breaking this expression into cases. Of the \( r \) textbooks, Shawn could choose 0 to be math textbooks, 1 to be a math textbook, or 2, 3, etc. (the remainder being made up of CS textbooks). We let this value be \( k \). That is, \( k \) is the number of math textbooks and \( r-k \) is the number of CS textbooks Shawn chooses. So we solve this problem in 2 steps, for each \( k \):

**Step 1:** Shawn chooses \( k \) of the \( m \) math textbooks.

**Step 2:** Shawn chooses \( r-k \) of the \( n \) CS textbooks.

We again apply the definition of the combination to find that there are \( \binom{m}{k} \) ways to perform Step 1 and \( \binom{n}{r-k} \) ways to perform Step 2. With the multiplication rule, we have \( \binom{m}{k} \binom{n}{r-k} \). Then we sum over all possible values of \( k \), which range over all integer values from 0 to \( r \) (Shawn could pick any value from 0 to \( r \) of the \( r \) textbooks to be math books):

\[
\sum_{k=0}^{r} \binom{m}{k} \binom{n}{r-k}
\]

This is exactly the expression on the RHS. Thus, as we have solved the same problem in two valid ways to get the expressions on the LHS and RHS respectively, we have a combinatorial proof.

15. Recall (and remember!) that the sum of the squares of the first \( n \) positive integers is given by the following formula: 

\[1^2 + 2^2 + \cdots + (n-1)^2 + n^2 = n(n+1)(2n+1)/6.\]

(a) Using only the formula above, (no credit in part (a) for proof by induction, see part (b)), derive the following formula for the sum of the squares of first \( m \) odd positive integers. Show your work.

\[1^2 + 3^2 + 5^2 + \cdots + (2m-3)^2 + (2m-1)^2 = \frac{m(4m^2-1)}{3}\]
(b) Now prove by induction the formula from part (a).

**Answer**

(a) Let \( D(m) = \sum_{k=1}^{m} (2k-1)^2 \). To this we add \( E(m) = \sum_{k=1}^{m} (2k)^2 \) and we obtain \( T(m) = \sum_{i=1}^{2m} i^2 \). By the formula, \( T(m) = 2m(2m+1)(4m+1)/6 \). Meanwhile, \( E(m) = \sum_{k=1}^{m} (2k)^2 = 2^2 \sum_{k=1}^{m} k^2 = 2^2 \cdot m(m+1)(2m+1)/6 \) (using again the formula). Hence

\[
D(m) = T(m) - E(m) = \frac{2m(2m+1)(4m+1)}{6} - \frac{4m(m+1)(2m+1)}{6} = \frac{2m(2m+1)(4m+1 - 2(m+1))}{6} = \frac{2m(2m+1)(2m-1)}{6} = \frac{m(4m^2-1)}{3}
\]

(b) (BASE CASE) \( m = 1 \). We have:

\[
\frac{1 \cdot (4 \cdot 1^2 - 1)}{3} = \frac{4 \cdot 1 - 1}{3} = \frac{3}{3} = 1 = 1^2 \checkmark
\]

(INDUCTION STEP) Let \( k \) be an arbitrary integer. Assume that (IH) \( 1^2 + 3^2 + \cdots + (2k-3)^2 + (2k-1)^2 = \frac{k(4k^2-1)}{3} \). In other words, the sum of the squares of the first \( k \) odd integers is equal to \( \frac{k(4k^2-1)}{3} \).

We wish to show that the claim holds for \( n = k+1 \), i.e. the sum of the squares of the first \( k+1 \) odd integers is equal to \( \frac{(k+1)(4(k+1)^2-1)}{3} \). We have:

\[
\sum_{i=1}^{k+1} (2i-1)^2 = \sum_{i=0}^{k} (2i-1)^2 + (2(k+1) - 1)^2 = \frac{k(4k^2-1)}{3} + (2k+1)^2 = \frac{k(4k^2-1)}{3} + 4k^2 + 4k + 1 = \frac{4k^3 - k + 12k^2 + 12k + 3}{3} = \frac{k(4k^2 + 8k + 3) + (4k^2 + 8k + 3)}{3} = \frac{(k+1)(4(k+1)^2-1)}{3}
\]

as desired. Thus, we have shown our claim is true when \( n = k + 1 \), concluding our Induction Step and completing our proof.
16. Bob is recycling a set $B$ of $m \geq 1$ distinguishable (he likes variety) bottles $B = \{b_1, \ldots, b_m\}$ in a facility that has a set $D$ of $n \geq 2$ distinguishable drums, $D = \{d_1, \ldots, d_n\}$. When Bob shows up all the drums are empty. Each drum is large enough to hold by itself all of Bob’s $m$ bottles. We call a deposit a way of placing the bottles in the drums, i.e., a function $t : B \to D$. Each deposit may leave some drums (maybe none) empty. Let empty($t$) be the set consisting of all the drums that are left empty by deposit $t$. (Note that it might be the case that empty($t$) = $\emptyset$, depending on $m, n$ and $t$.) Assume $m \geq n$ and prove that there exist two different deposits, $t_1$ and $t_2$ such that empty($t_1$) = empty($t_2$).

**Answer**

Note that there are $n^m$ ways to put $m$ bottles into $n$ drums (for each bottle, there are $n$ possible drums to deposit it into). Then, there are $n^m$ deposits: we will consider these the pigeons.

Observe that each deposit leaves some set of drums empty; by definition, the set that a deposit $t$ leaves empty is empty($t$) $\subseteq D$. Now we ask: what are the subsets of $D$ that can be empty($t$) for some deposit $t$?

To count these, we recognize that the only subset of $D$ that cannot be empty($t$) for some deposit $t$ is $D$ itself. (If every drum is empty, and $m \geq 1$ as required, where do these bottles go?)

Let’s prove that $E = 2^D \setminus \{D\}$ is exactly the set of all subsets of $D$ that can be empty($t$) for some deposit $t$. Note that since $D \notin E$ by definition, every element in $E$ contains at least zero and at most $n - 1$ drums (there is no other subset of $D$ that contains all $n$ elements of $D$). Now, consider an arbitrary set $F \in E$. For a deposit $t$ to generate empty($t$) = $F$, observe that $t$ must place at least one bottle $b_1, b_2, \ldots, b_n$ into every drum in $D \setminus F$. That is, filling every drum that is not in $F$ will yield an empty set of drums that is exactly $F$. Since $m \geq n$, and since the size of $D \setminus F$ is strictly positive and less than $n$ (since $0 \leq |F| \leq n - 1$ and $|D| = n$), we can construct such a deposit by following this procedure: deposit one bottle in each drum in the set, then deposit the remainder of the bottles into the drums at random. Now note that there is no deposit $t$ for which empty($t$) is equal to a set that is not in $E$. This can be seen by observing that those drums that are left empty by a deposit must come from the set $D$; then, empty($t$) must be an element of $2^D$. As argued previously, itself cannot be left empty, else there must be at least one bottle that is not deposited.

Then, there are $|2^D \setminus \{D\}|$ many subsets of $D$ that empty($t$) can take on for a deposit $t$. By calculation, $|2^D \setminus \{D\}| = 2^n - 1$. These will be our pigeonholes.

By the PHP, there must be a subset of $D$ that is the set of drums left empty for at least $\left\lceil \frac{n^m}{2^n - 1} \right\rceil$ many distinct deposits.

To show that $\left\lceil \frac{n^m}{2^n - 1} \right\rceil \geq 2$, note that $n^m \geq 2^m \geq 2^n > 2^n - 1$, because $m \geq n$ and $n \geq 2$ as given. Then, $\left\lceil \frac{n^m}{2^n - 1} \right\rceil \geq 2$, and there must be at least two distinct deposits that admit the same set of empty drums, as desired.

**Alternate Solution:** Observe that this question can also be answered by constructing two deposits $t_1, t_2$ that yield empty($t_1$) = empty($t_2$).

First, fix the drums in some order $d_1, d_2, \ldots, d_n$. Then, fix the bottles in some order $b_1, b_2, \ldots, b_m$. We define deposit $t_1$ by placing each bottle $b_i$ into drum $d_i$ for $1 \leq i \leq n$. For $i > n$, place $b_i$ in drum
Observe that since there is at least one bottle in each drum, \( \text{empty}(t_1) = \emptyset \).

Now, consider the deposit in generated by taking the order of drums \( d_1, d_2, \ldots, d_n \) and the order of bottles \( b_1, b_2, \ldots, b_m \) as above and placing bottle \( b_i \) into drum \( d_i \) for \( 2 \leq i \leq n-1 \), placing bottle \( b_1 \) in drum \( d_n \), and all bottles \( b_i \) into drum \( d_1 \) for \( i \geq n \). Call this deposit \( t_2 \). Observe that each drum necessarily has at least one bottle in it, since \( m \geq n \), and that \( t_2 \neq t_1 \), since the drums and bottles are distinct and bottle \( b_1 \) is in a different drum in each deposit (also, there are \( n \geq 2 \) drums by the bounds given in the problem, so these drums are necessarily distinct). From this, we can conclude that \( \text{empty}(t_2) = \emptyset \).

Then, we have constructed two distinct deposits \( t_1, t_2 \) such that \( \text{empty}(t_1) = \text{empty}(t_2) \), as desired.

17. Let \( A, B \) be two sets such that \( |A \cup B| = 12 \) and \( |A \cap B| = 8 \). Prove that \( 96 \leq |A \times B| \leq 100 \).

**Answer**

Let \( x = |A| \) and \( y = |B| \). By Inclusion-Exclusion \( 12 = |A \cup B| = x + y - |A \cap B| = x + y - 8 \). Hence, \( x + y = 20 \) thus \( y = 20 - x \).

By the multiplication rule, \( |A \times B| = |A| \cdot |B| = xy = x(20 - x) \). So we need to show that \( 96 \leq x(20 - x) \leq 100 \).

\( x(20 - x) \leq 100 \) is equivalent to \( (x - 10)^2 \geq 0 \) which is obvious.

\( 96 \leq x(20 - x) \) is equivalent to \( (x - 8)(x - 12) \leq 0 \). But \( A \cap B \subseteq A \subseteq A \cup B \) hence \( 8 \leq x \leq 12 \). It follows that \( (x - 8)(x - 12) \leq 0 \).

18. Consider the recurrence relation

\[
a_0 = 0 \quad a_1 = 1 \quad a_n = 2a_{n-1} - a_{n-2} + 1 \quad (\text{for } n \geq 2)
\]

Express \( a_n \) as a polynomial in \( n \). (Hint: use the telescopic trick twice.) Then prove by induction the result you obtained.

**Answer**

If we write the recurrence relation for \( n = 2, \ldots, n \), add the equations and cancel terms using the telescopic trick,

\[
\begin{align*}
\alpha_2 &= 2a_1 - a_0 + 1 \\
\alpha_3 &= 2a_2 - a_1 + 1 \\
\alpha_4 &= 2a_3 - \alpha_2 + 1 \\
\alpha_5 &= 2a_4 - \alpha_3 + 1 \\
&\quad \vdots \\
a_{n-1} &= 2a_{n-2} - a_{n-3} + 1 \\
a_n &= 2a_{n-1} - a_{n-2} + 1
\end{align*}
\]

we obtain that

\[
\begin{align*}
a_n + a_{n-1} &= 2a_{n-1} + 2a_1 - a_1 - a_0 + (n - 1) \\
a_n &= a_{n-1} + a_1 - a_0 + (n - 1) \\
a_n &= a_{n-1} + n
\end{align*}
\]
Applying the same telescopic trick to this recurrence:

\[
\begin{align*}
\alpha_1 &= a_0 + 1 \\
\alpha_2 &= \alpha_1 + 2 \\
\alpha_3 &= \alpha_2 + 3 \\
&\quad \vdots \\
\alpha_{n-1} &= \alpha_{n-2} + (n - 1) \\
\alpha_n &= \alpha_{n-1} + n
\end{align*}
\]

we get that

\[
a_n = a_0 + \sum_{i=1}^{n} i = \frac{n(n + 1)}{2}
\]

Therefore, \(a_n = \frac{n(n+1)}{2}\).

19. Consider 33 distinct boolean expressions in exactly two variables. Prove that 3 or more of them must be logically equivalent.

**Answer**

We wish to show that given 33 distinct boolean expressions, 3 or more of them have the same truth table (meaning that each group is logically equivalent).

Observe that a truth table in two variables is a function \(tt : S \to \{T, F\}\), where \(S\) is the set of sequences of length 2 formed from \(\{T, F\}\). \(|S| = 2^2 = 4\). Thus, there are \(2^{|S|} = 2^4 = 16\) distinct truth tables in two variables.

Now, by GPHP, since \(33 = 2 \cdot 16 + 1\), 3 or more of the expressions must have the same truth table.

20. In an All-Milky Way course the students receive their graded homeworks consisting of \(n \geq 2\) problems, where each problem is given a score between 0 and \(m \geq 1\). Assume that there are enough students (hence the galaxy-wide offering :) such that each set of possible scores on each problem is represented in the scores received by the students. For any two students \(a\) and \(b\) define \(\text{Same}(a, b)\) to be the set of all homework problems on which \(a\) and \(b\) got the same scores.

Use the Pigeonhole Principle to prove that there exist four students, \(a, b, c, d\) such that

- \(\text{Same}(a, b) = \text{Same}(c, d)\), and
- \(a \neq b\), and
- \(c \neq d\), and
- \(a \neq c\) OR \(b \neq d\)

**Answer**

The four students form two ordered pairs, each of two distinct students, \((a, b)\) and \((c, d)\). Thus it comes down to proving that there exist two distinct such ordered pairs such that \(\text{Same}(a, b) = \text{Same}(c, d)\).

How many ordered pairs of distinct students are there? Since we can assume that there are enough students so that every possible scoring occurs, this number is at least as big as the number of ordered pairs of distinct scorings.
It helps to realize that a scoring is a function with domain \([1..n]\) and codomain \([0..m]\). So there are \((m + 1)^n\) scorings. For each such scoring there are \((m + 1)^n - 1\) that differ from it. So there are \((m + 1)^n((m + 1)^n - 1)\) ordered pairs of distinct scorings, and therefore at least this many pairs of distinct students.

Now, with each ordered pair \((a, b)\) of students we associate the set \(\text{Same}(a, b)\) of all the problems on which the students \(a\) and \(b\) received the same score. There are at most \(2^n\) such sets (since there are \(2^n\) subsets of the \(n\) problems).

To apply the pigeonhole principle let the pairs of distinct students be the pigeons, the sets of problems be the pigeonholes and \(\text{Same}\) be the function that assigns pigeons to pigeonholes.

Our proof will be done if we can show that there are strictly more pigeons than pigeonholes. Indeed, since \(m \geq 1\) and \(2^n - 1 \geq 2 - 1 > 1\) we have \((m + 1)^n((m + 1)^n - 1) \geq 2^n(2^n - 1) > 2^n\).

**21.** For each statement below, decide whether it is TRUE or FALSE. In each case attach a very brief explanation of your answer.

(a) Let \((\Omega, \Pr)\) be a probability space with three outcomes. Let \(E, F\) be two nonempty events in this space such that \(\Pr[E \cup F] = \Pr[E] + \Pr[F]\). Then \(E \cap F = \emptyset\).

(b) Let \(A, B, C\) be three events of non-zero probability in a probability space \((\Omega, \mathcal{P})\). If \(A \cap B = B \cap C\), \(A \perp B\), and \(B \perp C\) then \(\Pr[A] = \Pr[C]\).

(c) If a probability space has an event of probability \(2/3\) then it must have some outcome of probability at most \(1/3\), true or false?

(d) Let \(E, F\) be two events in a finite probability space. If \(|E| = |F|\) then \(\Pr[E] = \Pr[F]\), true or false?

(e) If \(E, F\) are two events in a finite probability space such that \(\Pr[E \cap F] > 0\) then \(E\) and \(F\) can be disjoint, true or false?

(f) Let \(A, B\) be events in a finite probability space such that \(\Pr[A] = 1/4\) and \(\Pr[A \cup B] = 1/2\). Then, \(1/4 \leq \Pr[B] \leq 1/2\), true or false?

(g) For any three events \(E, F, G\) in the same probability space. if \(E \perp F\) and \(F \perp G\) then \(E \perp G\).

**ANSWER**

(a) FALSE.

Consider \(\Omega = \{w_1, w_2, w_3\}\) where \(\Pr[w_1] = \Pr[w_2] = 1/2\) and \(\Pr[w_3] = 0\). Define events \(E = \{w_1, w_3\}\) and \(F = \{w_2, w_3\}\). We have

\[
\Pr[E \cup F] = \Pr[\Omega] = 1
\]

and

\[
\Pr[E] + \Pr[F] = (1/2 + 0) + (1/2 + 0) = 1 = \Pr[E \cup F]
\]

However \(E \cap F = \{w_3\} \neq \emptyset\).
(b) TRUE.
We first observe that, as $A$ and $B$ are independent, we have that
\[ \Pr[A] \cdot \Pr[B] = \Pr[A \cap B] \]
However, we also have that $A \cap B = B \cap C$, meaning $\Pr[A \cap B] = \Pr[B \cap C]$. Furthermore, since $B$ and $C$ are independent, we have that:
\[ \Pr[B] \cdot \Pr[C] = \Pr[B \cap C] \]
Combining these facts, we have that:
\[ \Pr[A] \cdot \Pr[B] = \Pr[B] \cdot \Pr[C] \]
Since $A, B, C$ have non-zero probabilities, we can conclude that
\[ \Pr[A] = \Pr[C] \]

(c) TRUE.
Let $A$ be the event such that $\Pr[A] = \frac{2}{3}$. Then, we know that the complement of $A$, $\overline{A}$, has probability
\[ \Pr[\overline{A}] = \frac{1}{3} \]
Since $\Pr[\overline{A}] \neq 0$, $\overline{A} \neq \emptyset$, meaning $\overline{A}$ contains at least one outcome. Let $w$ be an outcome in $\overline{A}$. We know that:
\[ \Pr[w] \leq \Pr[\overline{A}] = \frac{1}{3} \]
Thus, $w$ is an outcome with probability at most $\frac{1}{3}$.

(d) FALSE.
We can find a counterexample by defining a non-uniform probability space and letting $E$ and $F$ be sets of single outcomes with different probabilities. For example, consider the roll of two indistinguishable dice. Let $E$ be the event that the roll results in two 6’s and let $F$ be the event that the roll results in one 5 and one 6. $|E| = |F| = 1$, but $\Pr[E] = \frac{1}{36}$ and $\Pr[F] = \frac{1}{18}$.

(e) FALSE.
If $\Pr[E \cap F] > 0$, there exists some outcome $\omega \in E \cap F$ that occurs with a positive probability, so $E \cap F \neq \emptyset$.

(f) TRUE.
By the Principle of Inclusion-Exclusion,
\[ \Pr[A \cup B] = \Pr[A] + \Pr[B] - \Pr[A \cap B] \]
Substituting, we have
\[ \frac{1}{2} = \frac{1}{4} + \Pr[B] - \Pr[A \cap B] \]
Hence \( \Pr B = \frac{1}{4} + \Pr A \cap B \). Since probabilities must be non-negative, \( \Pr [A \cap B] \geq 0 \); thus
\[
\Pr [B] \geq \frac{1}{4}
\]
Further, since \( B \subseteq A \cup B \), we know that
\[
\Pr [B] \geq \Pr [A \cup B] = \frac{1}{2}
\]
Thus, \( \frac{1}{4} \leq \Pr [B] \leq \frac{1}{2} \).

(g) FALSE. Take \( E, F \) such that \( E \perp F \), \( E \) such that \( \Pr [E] = \frac{1}{2} \), and \( G = E \). Obviously, since \( E \perp F \) and \( G = E \), then \( G \perp F \). However, let’s look at if \( E \perp G \). We can say that this is not the case \( (E \not\perp G) \) because \( \Pr [E \cap G] = \Pr [E \cap E] = \Pr [E] = \frac{1}{2} \) while \( \Pr [E] \cdot \Pr [G] = (1/2)(1/2) = 1/4 \).

22. Let \( A, B, C \) be three events in the same probability space such that \( A \subseteq B \), \( A \subseteq C \), \( B \perp C \), and \( \Pr [A] = 1 \). Prove that \( \Pr [A \cap B \cap C] = \Pr [A] \Pr [B] \Pr [C] \).

**Answer**

Since \( A \subseteq B \) and \( A \subseteq C \), we know that \( A \subseteq B \cap C \) (one way to reason about this is to observe that \( A \cap C \subseteq B \cap C \), and plug in \( A = A \cap C \)). Therefore, \( A \cap B \cap C = A \).

Moreover, by monotonicity of probability, \( A \subseteq B \) implies \( \Pr [A] \leq \Pr [B] \). Since \( 1 = \Pr [A] \leq \Pr [B] \leq 1 \) we have \( 1 \leq \Pr [B] \leq 1 \), which means that we must have \( \Pr [B] = 1 \).

Similarly, we can show that \( \Pr [C] = 1 \).

Therefore, \( \Pr [A \cap B \cap C] = \Pr [A] = 1 = 1 \cdot 1 \cdot 1 = \Pr [A] \Pr [B] \Pr [C] \).

23. Let \( E, F \) be two events in a finite probability space such that \( \Pr [E \cap F] > 0 \). Prove that \( \Pr [E \setminus F] + \Pr [F \setminus E] < \Pr [E \cup F] \).

**Answer**

Note that we can see the union of two events is the same as the part of the events unique to either event combined with the parts of the event shared between the two and thus: \( (E \setminus F) \cup (E \cap F) \cup (F \setminus E) = E \cup F \). By the Sum Rule, since the LHS sets are pairwise disjoint:
\[
\Pr [E \setminus F] + \Pr [E \cap F] + \Pr [F \setminus E] = \Pr [E \cup F]
\]
Therefore \( \Pr [E \setminus F] + \Pr [F \setminus E] = \Pr [E \cup F] - \Pr [E \cap F] < \Pr [E \cup F] \).