CIS 160 - Spring 2019 (Instructors Greenberg and Tannen)

Midterm 2 Review
Posted Monday March 18

1 Readings

STUDY IN-DEPTH... ...the posted notes for lectures 9-15 including the supplements.

STUDY IN-DEPTH... ...the posted guides for recitations 5-7.

STUDY IN-DEPTH... ...the posted solutions to homeworks 4-6. Compare with your own solutions.

STUDY IN-DEPTH... ...the solutions to the mock exam and the additional problems contained in this document, to be posted Thursday March 21, in the afternoon. Until then, try very hard to solve these on your own.

2 Memorize!

Find and memorize formulas:

- For the sum of a geometric progression.
- For the sum of the integers in \([1..n]\).
- For the sum of the squares of the integers in \([1..n]\).

3 Mock Exam (60 minutes for 120 points)

1. (30 pts)

For each statement below, decide whether it is TRUE or FALSE and circle the right one. In each case attach a very brief explanation of your answer.

(a) Assume that \(B\) is a set with 7 elements and that \(A\) is a set with 15 elements. Then, for any function \(f : A \rightarrow B\) there exist at least 3 distinct elements of \(A\) that are mapped by \(f\) to the same element of \(B\).

(b) For any \(A, B, C\) non-empty finite sets, let \(m = |A| + |B| + |C| - |A \cup B \cup C|\) and \(n = |A \cap B| + |B \cap C| + |C \cap A|\). Then \(m > n\).

(c) If \(X_1\) and \(X_2\) are Bernoulli random variables with \(\Pr[X_1 = 1] = 1/2\) and \(\Pr[X_2 = 1] = 1/3\) then \(E[X_1 - X_2] = 0\).

(d) For any two events \(A, B\) in the same probability space \((\Omega, \Pr)\) such that \(\Pr[B] \neq 0\) we have \(\Pr[A \cup B | B] = 1\).
(e) If the set $A$ has $n$ elements then there are $n!$ injective functions with domain $A$ and codomain $A$.

(f) The Fibonacci number $F_{100}$ is even.

2. (15pts) My 6th grade teacher of Russian was unable to pay attention to what we were answering and it appeared to us that he was assigning grades completely randomly. Let’s assume that his grading rubric consisted of tossing a fair coin six times, counting the number $k$ of heads and assigning the grade $4 + k$ (our grades were in the 1-10 range).

(a) What was the probability that I would get a 10?

(b) What was the probability of the following event: “my grade was divisible by 4 or (non-exclusive or!) it was bigger than or equal to Lady Gaga’s shoe size (a 6)”?

3. (20pts)

A fair coin is flipped twice. Let $(Ω, Pr)$ be the resulting probability space. Let $X_H$ be random variable defined on $Ω$ that returns the number of heads observed and $X_T$ similarly the number of tails observed.

(a) Describe the probability space $(Ω, Pr)$. That is, list the outcomes and their probabilities.

(b) Show that the random variable $Z$ defined by $∀ w ∈ Ω \ Z(w) = X_H(w) \cdot X_T(w)$ is a Bernoulli random variable and find its probability of success.

(c) Show that $\mathbb{E}[Z] \neq \mathbb{E}[X_H] \mathbb{E}[X_T]$.

4. (25pts)

Let $R_n = \sum_{k=1}^{2n} (-1)^{k+1} k$ for $n ≥ 1$.

(a) Compute $R_1, R_2, R_3$. Guess a simple way to express $R_n$ in terms of $n$. Prove your guess by induction.

(b) Prove by induction that for all $n ≥ 1$ we have

$$1 + 3 + 5 + \cdots + (2n - 1) = n^2$$

(c) Use the identity in part (b) and other identities that you were supposed to memorize to prove the identity in part (a).

5. (20pts)

Alice is splitting atoms in Wonderland. She has two elements, madium and hatium. Every second, every madium atom splits into two madium atoms and one hatium atom, and every hatium atom splits into two hatium atoms. Assume that Alice starts in second 1 with one madium atom and one hatium atom. Then, for example, in second 2 she will have 2 madium atoms and 3 hatium atoms. Et caetera.

Guess, as an expression in terms of $n$, the number of hatium atoms that Alice will have in second $n$, and prove your answer.
6. (10pts)

A lottery urn contains \( n \geq 2 \) distinct balls labeled with the numbers 1, \ldots, \( n \). You extract two distinct balls from the urn. Suppose they are labeled \( i \) and \( j \). You compute \( i + j \) and write down the answer on a piece of paper. Then you put the two balls back.

You repeat this \( m \) times. What is the smallest value of \( m \) that ensures (no probabilities in this problem!) that you will end up writing the same number at least twice on the piece of paper. Prove your answer.

4 Additional Problems

1. Recall (and remember!) that the sum of the squares of the first \( n \) positive integers is given by the following formula:
\[
1^2 + 2^2 + \cdots + (n-1)^2 + n^2 = n(n+1)(2n+1)/6.
\]

(a) Using only the formula above, (no credit in part (a) for proof by induction, see part (b)), derive the following formula for the sum of the squares of first \( m \) odd positive integers. Show your work.
\[
1^2 + 3^2 + 5^2 + \cdots + (2m-3)^2 + (2m-1)^2 = \frac{m(4m^2-1)}{3}
\]

(b) Now prove by induction the formula from part (a).

2. Bob is recycling a set \( B \) of \( m \geq 1 \) distinguishable (he likes variety) bottles \( B = \{b_1, \ldots, b_m\} \) in a facility that has a set \( D \) of \( n \geq 2 \) distinguishable drums, \( D = \{d_1, \ldots, d_n\} \). When Bob shows up all the drums are empty. Each drum is large enough to hold by itself all of Bob’s \( m \) bottles. We call a deposit a way of placing the bottles in the drums, i.e., a function \( t : B \to D \). Each deposit may leave some drums (maybe none) empty. Let \( \text{empty}(t) \) be the set consisting of all the drums that are left empty by deposit \( t \). (Note that it might be the case that \( \text{empty}(t) = \emptyset \), depending on \( m, n \) and \( t \).)

Assume \( m \geq n \) and prove that there exist two different deposits, \( t_1 \) and \( t_2 \) such that \( \text{empty}(t_1) = \text{empty}(t_2) \).

3. Let \( A, B \) be two sets such that \(|A \cup B| = 12\) and \(|A \cap B| = 8\). Prove that \( 96 \leq |A \times B| \leq 100 \).

4. Consider the recurrence relation
\[
a_0 = 0 \quad a_1 = 1 \quad a_n = 2a_{n-1} - a_{n-2} + 1 \quad \text{for } n \geq 2
\]

Express \( a_n \) as a polynomial in \( n \). (Hint: use the telescopic trick twice.) Then prove by induction the result you obtained.

5. Prove that any positive integer can be expressed as the sum of distinct Fibonacci numbers.

6. Consider 33 distinct boolean expressions in exactly two variables. Prove that 3 or more of them must be logically equivalent.
7. In an All-Milky Way course the students receive their graded homeworks consisting of $n \geq 2$ problems, where each problem is given a score between 0 and $m \geq 1$. Assume that there are enough students (hence the galaxy-wide offering :) such that each set of possible scores on each problem is represented in the scores received by the students. For any two students $a$ and $b$ define $\text{Same}(a, b)$ to be the set of all homework problems on which $a$ and $b$ got the same scores.

Use the Pigeonhole Principle to prove that there exist four students, $a, b, c, d$ such that

- $\text{Same}(a, b) = \text{Same}(c, d)$, and
- $a \neq b$, and
- $c \neq d$, and
- $a \neq c$ OR $b \neq d$

8. (15pts) Alice has a strange coin that shows the number 3 on one side and the number 5 on the other. Still, the coin is fair. Bob has strange die that shows the numbers 5, 6, 7, 8, 9, 10 on its six faces. Still, the die is fair. Alice flips the coin and, independently, Bob rolls the die. What is the probability that the number on the die is divisible by the number on the coin?

9. (25pts) Alice has a fair coin that shows the number 2 on one side and the number 3 on the other. Bob has a fair tetrahedral die (a tetradie) that shows the numbers 1, 2, 3 and 4 on its four faces. They play the following game:

- Alice flips the coin showing the number $a$ and, independently, Bob rolls the tetradie showing the number $b$
- If $a > b$ then Alice wins and Bob pays Alice $a - b$ dollars. If $a = b$ then it’s a tie and no money changes hands. If $b > a$ then Bob wins and Alice pays Bob $b - a$ dollars.

(a) Draw the tree of possibilities for a single game.
(b) Compute the probability that Alice wins a single game.
(c) Suppose that Alice and Bob play the game 3 times in a row, independently. Assume that Alice starts with 10 dollars. Let $Z$ be the random variable that returns the amount of dollars that Alice has after these 3 games. Compute $E[Z]$.

10. For each statement below, decide whether it is TRUE or FALSE In each case attach a very brief explanation of your answer.

(a) Let $(\Omega, \Pr)$ be a probability space with three outcomes. Let $E, F$ be two nonempty events in this space such that $\Pr[E \cup F] = \Pr[E] + \Pr[F]$. Then $E \cap F = \emptyset$.
(b) Let $A, B, C$ be three events of non-zero probability in a probability space $(\Omega, P)$. If $A \cap B = B \cap C$, $A \perp B$, and $B \perp C$ then $\Pr[A] = \Pr[C]$.
(c) If a probability space has an event of probability $2/3$ then it must have some outcome of probability at most $1/3$, TRUE or FALSE?
(d) Let $A, B$ be events in a probability space such that $\Pr[A] = 0$ and $\Pr[B] \neq 0$. Then, $\Pr[A \mid B] = 0$, true or false?
(e) For any probability space \((\Omega, P)\) and any event \(A \subseteq \Omega\) such that \(\Pr[A] \neq 0\) we have \(\Pr[\Omega \mid A] = \Pr[A \mid \Omega]\), true or false?

(f) Let \(E, F\) be two events in a finite probability space. If \(|E| = |F|\) then \(\Pr[E] = \Pr[F]\), true or false?

(g) If \(E, F\) are two events in a finite probability space such that \(\Pr[E \cap F] > 0\) then \(E\) and \(F\) can be disjoint, true or false?

(h) Let \(A, B\) be events in a finite probability space such that \(\Pr[A] = 1/4\) and \(\Pr[A \cup B] = 1/2\). Then, \(1/4 \leq \Pr[B] \leq 1/2\), true or false?

(i) For any three events \(E, F, G\) in the same probability space. if \(E \perp F\) and \(F \perp G\) then \(E \perp G\).

11. Let \(A, B, C\) be three events in the same probability space such that \(A \subseteq B, A \subseteq C, B \perp C,\) and \(\Pr[A] = 1\). Prove that \(\Pr[A \cap B \cap C] = \Pr[A] \Pr[B] \Pr[C]\).

12. Let \(E, F\) be two events in a finite probability space such that \(\Pr[E \cap F] > 0\). Prove that \(\Pr[E \setminus F] + \Pr[F \setminus E] < \Pr[E \cup F]\).

13. Alice has an urn with three marbles labeled 1, 2, and 3. Each of the marbles is equally likely to be extracted. Bob has a separate, similar urn. They play the following game of chance:

   1. Alice extracts a marble from her urn and obtains \(a \in \{1, 2, 3\}\).
   2. Independently, Bob extracts a marble from his urn and obtains \(b \in \{1, 2, 3\}\).
   3. If \(a > b\) then Alice wins. If \(b > a\) then Bob wins. If \(a = b\) they flip a fair coin and if the coin shows heads, Alice wins. If the coin shows tails, Bob wins.

In the calculations below, do not spend time on the arithmetic. It’s OK to leave your results as products and fractions.

(a) Draw the “tree of possibilities” diagram for this game, with all the outcomes and their probabilities.

(b) Compute the probability that the game was decided by a coin flip.

(c) Compute the conditional probability that Alice wins, knowing that Bob extracted the marble labeled 2.

(d) Alice and Bob put bets on the game. If Alice wins without a coin flip Bob pays her 2$. If Alice wins with a coin flip then Bob pays her 18. If Bob wins then Alice pays him 1.5$. What is Alice’s expected monetary win/loss (wins are positive, losses are negative) after \(n\) such games?

14. A fair coin is flipped \(2n\) times \((n \geq 1)\), independently. Let \(X_H\) the random variable that returns the number of heads that occurred and \(X_T\) the random variable that returns the number of tails that occurred. Compute \(P(X_H > X_T)\).

15. For each statement below, decide whether it is TRUE or FALSE and circle the right one. In each case attach a very brief explanation of your answer.

(a) If \(n \geq 1\) then \(1 + 3 + \cdots + 3^n < \left(\frac{3}{2}\right) 3^n\).
(b) Let $A, B$ be events in the same probability space and let $I_A, I_B$ be their indicator random variables. If $E(I_A + I_B) = 1$ then $P(A) = P(B)$.

16. Let $(\Omega, P)$ be a probability space and let $X$ be a random variable defined on $\Omega$ such that $\text{Val}(X) = \{a, b\}$ where $a < b$. We also denote $\mu = E(X)$.

(a) Express $P(X \leq (a + b)/2)$ in terms of $a, b$ and $\mu$.

(b) Let $a = -1$ and $b = 1$. Show that if $E(X) = 0$ then there exists an event $A \subseteq \Omega$ such that $P(A) = 1/2$.

17. Weird Al (WAl) is playing with his coins. The game uses two fair coins and one urn. The result of the game is one of $H$ (heads) or $T$ (tails) and is determined as follows:

- WAl places both coins in the urn.
- WAl reaches inside the urn and (a) with probability $2/3$ WAl grabs one of the coins and tosses it, OR (b) with probability $1/3$ WAl grabs both coins, then tosses them separately in some order (doesn’t matter which order).
- If WAl has tossed just one coin then whatever that coin shows is the result of the game. If WAl has tossed both coins then applying the weird $\otimes$ operation to what the two coins show is the result of the game, where $T \otimes T = T$, $T \otimes H = H$, $H \otimes T = H$, and $H \otimes H = T$.

(a) Draw the “tree of possibilities” diagram for WAl’s game.

(b) Calculate the probability that the result of the game is $H$.

(c) What simpler game could Weird Al play that would give him exactly the same odds?

18. Let $S$ be the probability space $(\Omega, P)$ with $\Omega = \{f : \mathbb{R} \to \mathbb{R} | f(x) = ax + b\}$ such that $a, b \in \mathbb{N}$, $1 \leq a \leq 10$, and $1 \leq b \leq 10$, and $P$ is the uniform probability distribution on $\Omega$. For each natural number $k$, Let $X_k$ be the random variable that takes on the value of $f(k)$.

(a) What is $E[X_5]$?

(b) What is $P(X_5 > 1)$?

(c) What is $E[X_k]$ in terms of $k$?

(d) Now consider the probability space $S' = (\frac{d\Omega}{dx}, P)$, where $\frac{d\Omega}{dx} = \{\frac{d}{dx}[f] | f \in \Omega\}$. Let $Y$ be the random variable that takes on $f(5)$. What is $E[Y]$?

19. You have a standard deck of 52 cards, from which you draw 13 cards, without replacement.

(a) Given that you drew the 4 of spades, what is the probability that all the other cards that you drew are aces, twos, threes, or fours?

(b) Define $S$ to be the number of spades you draw. What is $E[\lceil S \rceil]$?

(c) For what $s \in \text{Val}(S)$ do we have the maximum value of $\Pr[S = s]$?

(d) Suppose that the number of spades you have in your hand is equal to the number you found in part (c). What is the probability that the sum of their numerical values (letting $J = 11$, $Q = 12$, $K = 13$, $A = 1$) is odd?