Exam 2 Review

Posted Friday June 28

1 Readings

STUDY IN-DEPTH... ...the posted notes for lectures 8-15 including the supplements. (See below for details about lecture 8.)

STUDY IN-DEPTH... ...the posted guides for recitations 4 and 5.

STUDY IN-DEPTH... ...the posted solutions to homeworks 4 and 5. Compare with your own solutions.

STUDY IN-DEPTH... ...the solutions to the mock exam and the additional problems contained in this document will be posted on Sunday, June 30th, in the morning. Until then, try very hard to solve these on your own.

For this exam, you must review carefully some material from lecture 8, specifically everything about the Monty Hall problem.

2 Not cumulative?

Superficially, Exam 2 is not cumulative. Yet, you cannot solve most problems from part 2 of this course without knowing how to do proofs, knowing counting techniques, knowing the basics of probability and having an understanding of mathematical notation. All these were covered in lecture 1-8 so they do not appear on the readings list above. However, if you still have gaps in understanding the material from part 1 now is a good time to close them.

3 Memorize!

Find and memorize formulas:

- For the expectation and for the variance of a Bernoulli random variable.
- For the expectation and for the variance of a binomial random variable.

4 Mock Exam (120 minutes for 240 points)

1. (20 pts)
   
   (a) Draw a connected undirected graph with 6 nodes and exactly 2 cut edges.
   
   (b) Draw an acyclic undirected graph with 5 edges and 7 nodes.
(c) Draw a connected undirected graph in which every node has degree 3.
(d) Draw a digraph that has no sources and no sinks and has strictly more edges than vertices.

2. (35 pts)
Each of the parts in this problem is 5-10pts.
(a) Let $G = (V, E)$ be an undirected graph in which every node has degree 3. Prove that $|E|/3 = |V|/2$.
(b) Prove or disprove: if we add one edge to a tree between any two existing vertices the resulting graph cannot be bipartite.
(c) Let $X, Y, Z$ be random variables on the same probability space such that $Z = X + Y - XY$. Show that if $X$ and $Y$ are Bernoulli random variables, then $Z$ is also a Bernoulli random variable.
(d) Let $A, B$ be events in the same probability space and let $I_A, I_B$ be their indicator random variables. Prove that if $E(I_A + I_B) = 1$ then $P(A) = P(\overline{B})$.
(e) For any positive integer $n \geq 2$, if the complete bipartite graph $K_{2,n}$ has a cycle of length $n + 2$ then $n$ must be even.

3. (25pts)
A fair coin is flipped twice. Let $(\Omega, \Pr)$ be the resulting probability space. Let $X_H$ be random variable defined on $\Omega$ that returns the number of heads observed and $X_T$ similarly the number of tails observed.
(a) Describe the probability space $(\Omega, \Pr)$. That is, list the outcomes and their probabilities.
(b) Show that the random variable $Z$ defined by $\forall w \in \Omega \quad Z(w) = X_H(w) \cdot X_T(w)$ is a Bernoulli random variable and find its probability of success.
(c) Show that $E[Z] \neq E[X_H]E[X_T]$.

4. (20 pts) Let $B$ be a binomial random variable with parameters $p$ and $n$. Show that $R = n - B$ is also a binomial random variable and calculate its parameters.

5. (20 pts) Let $T$ be a tree with at least 3 vertices. Assume that every vertex in $T$ has either degree 3 or is a leaf. Let $L$ be the set of leaves of $T$ and let $R$ be the set of vertices in $T$ that have degree 3. Show that $|R| = |L| - 2$.

6. (20 pts) Prove by induction that any tree with at least 3 vertices must have at least one vertex of degree $\geq 2$. (Only proofs by induction will receive credit.)

7. (35pts) Alice has a fair coin that shows the number 2 on one side and the number 3 on the other. Bob has a fair tetrahedral die (a tetradie) that shows the numbers 1, 2, 3 and 4 on its four faces. They play the following game:
- Alice flips the coin showing the number $a$ and, independently, Bob rolls the tetradie showing the number $b$
• If $a > b$ then Alice wins and Bob pays Alice $a - b$ dollars. If $a = b$ then it’s a tie and no money changes hands. If $b > a$ then Bob wins and Alice pays Bob $b - a$ dollars.

(a) Draw the tree of possibilities for a single game.
(b) Compute the probability that Alice wins a single game.
(c) Suppose that Alice and Bob play the game 3 times in a row, independently. Assume that Alice starts with 10 dollars. Let $Z$ be the random variable that returns the amount of dollars that Alice has after these 3 games. Compute $E[Z]$.

8. (20 pts) Let $n \geq 3$ be a positive integer. Consider $K_{3,n}$, the complete bipartite graph with 3 red nodes and $n$ blue nodes.

(a) Consider a cycle of length 6 in $K_{3,n}$. How many blue nodes must such a cycle have? Explain your answer.
(b) Count the number of paths of length 3 in $K_{3,n}$.

9. (30 pts) Let’s call Peano-digraph a digraph in which every vertex has outdegree 1.

(a) Prove that any Peano-digraph that is strongly connected is, in fact, a directed cycle.
(b) Count the number of different Peano-digraphs whose set of vertices is $[1..n]$, where $n$ is a positive integer?

10. (15 pts) Let $G = (V, E)$ be an undirected graph with $|V| = n \geq 3$ vertices and satisfying the following property. For any $u, v, w \in V$, distinct vertices, at least one of these three is adjacent to the other two, that is,

$$v - u - w \text{ OR } u - v - w \text{ OR } u - w - v$$

(Note that it is also possible that the 3 vertices are pairwise adjacent.) Prove that $G$ has at least $\frac{n^2}{2} - n$ edges.

5 Additional Problems

1. For each statement below, decide whether it is TRUE or FALSE In each case attach a very brief explanation of your answer.

(a) Let $X$ be a Bernoulli random variable such that $\text{Var}[X] = 0.2 \cdot E[X]$. Then, the probability of success for $X$ is 0.4.
(b) There exists a random variable $X$ for which $E[X^2] < (E[X])^2$.
(c) Let $A, B$ be events in a probability space such that $\Pr[A] = 0$ and $\Pr[B] \neq 0$. Then, $\Pr[A \mid B] = 0$, true or false?
(d) For any probability space $(\Omega, P)$ and any event $A \subseteq \Omega$ such that $\Pr[A] \neq 0$ we have $\Pr[\Omega \mid A] = \Pr[A \mid \Omega]$, true or false?
(e) If $X_1$ and $X_2$ are Bernoulli random variables with $\Pr[X_1 = 1] = 1/2$ and $\Pr[X_2 = 1] = 1/3$ then $E[X_1 - X_2] = 0$. 

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(f) For any two events $A, B$ in the same probability space $(\Omega, \Pr)$ such that $\Pr[B] \neq 0$ we have $\Pr[A \cup B \mid B] = 1$.

(g) There exists an undirected graph $G$ with 4 vertices such that both $G$ and its complement, $\overline{G}$, are connected.

2. Alice has an urn with three marbles labeled 1, 2, and 3. Each of the marbles is equally likely to be extracted. Bob has a separate, similar urn. They play the following game of chance:

(1) Alice extracts a marble from her urn and obtains $a \in \{1, 2, 3\}$.
(2) Independently, Bob extracts a marble from his urn and obtains $b \in \{1, 2, 3\}$.
(3) If $a > b$ then Alice wins. If $b > a$ then Bob wins. If $a = b$ they flip a fair coin and if the coin shows heads, Alice wins. If the coin shows tails, Bob wins.

Solve the problems below. Please do not spend time on the arithmetic. It is OK to leave your results as products and fractions in your calculations.

(a) Draw the “tree of possibilities” diagram for this game, with all the outcomes and their probabilities.
(b) Compute the probability that the game was decided by a coin flip.
(c) Compute the conditional probability that Alice wins given that Bob extracts marble labeled 2.
(d) Alice and Bob put bets on the game. If Alice wins without a coin flip Bob pays her 2$. If Alice wins with a coin flip then Bob pays her 1$. If Bob wins then Alice pays him 1.5$. What is Alice’s expected monetary win/loss (wins are positive, losses are negative) after $n$ such games?

3. A fair coin is flipped $2n$ times ($n \geq 1$), independently. Let $X_H$ the random variable that returns the number of heads that occurred and $X_T$ the random variable that returns the number of tails that occurred. Compute $P(X_H > X_T)$.

4. Let $(\Omega, P)$ be a probability space and let $X$ be a random variable defined on $\Omega$ such that $\text{Val}(X) = \{a, b\}$ where $a < b$. We also denote $\mu = E(X)$.

(a) Express $P(X \leq (a + b)/2)$ in terms of $a, b$ and $\mu$.
(b) Let $a = -1$ and $b = 1$. Show that if $E(X) = 0$ then there exists an event $A \subseteq \Omega$ such that $P(A) = 1/2$.

5. Weird Al (WAl) is playing with his coins. The game uses two fair coins and one urn. The result of the game is one of $H$ (heads) or $T$ (tails) and is determined as follows:

- WAl places both coins in the urn.
- WAl reaches inside the urn and (a) with probability $2/3$ WAl grabs one of the coins and tosses it, OR (b) with probability $1/3$ WAl grabs both coins, then tosses them separately in some order (doesn’t matter which order).
- If WAl has tossed just one coin then whatever that coin shows is the result of the game. If WAl has tossed both coins then applying the weird $\otimes$ operation to what the two coins show is the result of the game, where $T \otimes T = T$, $T \otimes H = H$, $H \otimes T = H$, and $H \otimes H = T$. 


(a) Draw the “tree of possibilities” diagram for WAl’s game.
(b) Calculate the probability that the result of the game is $H$.
(c) What simpler game could Weird Al play that would give him exactly the same odds?

6. Consider $X$ and $Y$, two independent Bernoulli random variables defined on the same probability space. We are given $\Pr[X = 1] = 1/3$ and $\Pr[Y = 1] = 1/4$. Compute $E[(X + Y)^2]$.

7. For each statement below, decide whether it is TRUE or FALSE and circle the right one. In each case attach a *very brief* explanation of your answer.

(a) There exists a tree in which every node is a leaf.
(b) The complete bipartite graph $K_{5,5}$ has a cycle of length 5.
(c) There exists an undirected graph with 23 vertices such that 11 of them have degree 11 and 12 of them have degree 12.
(d) There exists a connected undirected graph with 100 vertices and 50 edges.
(e) A graph has an edge that is not a cut edge. Then it must have three edges that are not cut edges.
(f) There exists a tree with exactly 3 leaves, in which the length of a longest path is 1000.
(g) Consider an *undirected* graph with 3 or more vertices and with *exactly* 3 connected components. In order to make this into a connected graph, we must add at least 2 edges.

8. Let $G$ be a bipartite graph in which every connected component is a cycle.

(a) Draw the smallest (minimal number of vertices) such $G$. (Just the drawing, no need for explanation)
(b) Prove that, not just in the smallest, but in *any* such $G$ the number of red nodes is equal to the number of blue nodes.

9. Consider a connected graph $G = (V, E)$ such that $|E| = |V|$. Prove that $G$ contains *exactly* one cycle.

10. Define the complement of a string of bits $w$ of length $n \geq 1$ to be the string obtained by replacing all the 0’s in $w$ with 1 and all the 1’s with 0’s. Clearly, the complement of the complement of $w$ is $w$. Now construct an undirected graph whose nodes are all the strings of bits of length $n$ and such that there is an edge $u-v$ exactly when $v$ is the complement of $u$ (equivalently, when $u$ is the complement of $v$). Prove that the resulting graph is bipartite.

11. Let $X, Y, Z$ be three finite nonempty sets such that $X \cap Y = \emptyset$, $Z \cap Y = \emptyset$, $X \cap Z = \emptyset$ and denote $|X| = m$, $|Y| = n$, $|Z| = p$. Assume that $m < n < p$. Let also $f : X \rightarrow Y$ and $g : Y \rightarrow Z$ be two functions. Consider the undirected graph $G = (V, E)$ where $V = X \cup Y \cup Z$ and

$$E = \{ \{x, f(x)\} \mid x \in X\} \cup \{ \{y, g(y)\} \mid y \in Y\}$$

(a) What is $|V|$ and what is $|E|$ (in terms of $m, n, p$)?
(b) What is the maximum number of nodes of degree 0 that $G$ can have (in terms of $m, n, p$)?
12. An \( r \)-regular graph is a graph in which the degree of each vertex is exactly \( r \). Show that any 3-regular graph must have an even number of vertices and a number of edges divisible by 3.

13. For each statement below, decide whether it is TRUE or FALSE and circle the right one. In each case attach a very brief explanation of your answer.

(a) Let \( G \) be a DAG with \( n \geq 2 \) vertices and let the sequence \( \sigma \) be a topological sort of \( G \). If \( u \) appears before \( v \) in \( \sigma \) then there exists a directed path from \( u \) to \( v \) in \( G \).

(b) A strongly connected digraph with at least two nodes can have neither sources nor sinks.

14. For the three parts below, use the following definition: for any digraph \( G = (V,E) \) without self-loops and without cycles of length 2, define an undirected graph \( G^u = (V,E^u) \) that has the same vertices as \( G \) and moreover in \( G^u \) we have an edge \( v \rightarrow w \) whenever we have the edge \( v \rightarrow w \) or the edge \( w \rightarrow v \) in \( G \).

(a) If \( G \) is strongly connected then \( G^u \) is connected. Prove or disprove.

(b) If \( G \) is a DAG then \( G^u \) is acyclic. Prove or disprove.

(c) Prove that if \( G \) is a DAG in which every sink is reachable from every source then \( G^u \) is connected.

15. Recall the complete undirected graph on \( n \) vertices, \( K_n \). Prove that for any \( n \geq 4 \) it is possible to assign direction to each of the edges of \( K_n \) such that the resulting digraph has exactly \( n - 2 \) strongly connected components.

16. Let’s call a slug a DAG \( G = (V,E) \) with at least 4 vertices, \(|V| \geq 4\), and such that \( G \) has exactly one source \( r \) and exactly one sink \( s \).

(a) Draw two different slugs, both with 4 vertices, one of them with 3 edges and the other one with 5 edges.

(b) Prove that in any slug, for every node \( u \) that is not \( r \) or \( s \), there exists a directed path from \( r \) to \( s \) that passes through \( u \).

17. Let \( G = (V,E) \) be a connected graph with at least two distinct spanning trees.

(a) Prove that \(|E| \geq |V|\).

(b) Prove that the graph has at least three distinct spanning trees.