1. For each statement below, decide whether it is TRUE or FALSE and circle the right one. In each case attach a very brief explanation of your answer.

(a) Let $G$ be a DAG with $n \geq 2$ vertices and let the sequence $\sigma$ be a topological sort of $G$. If $u$ appears before $v$ in $\sigma$ then there exists a directed path from $u$ to $v$ in $G$.

Solution:
False  Counter example:

A topological sort would be 1423. Let $u = 1$ and $v = 4$. However, there is no path from 1 to 4.

(b) A strongly connected digraph with at least two nodes can have neither sources nor sinks.

Solution:
True. A strongly connected digraph with at least two nodes can have neither sources nor sinks. Assume for contradiction that there is source $v$. Then there must be no edges leading to $v$, and therefore no paths to $v$. The graph cannot be strongly connected, so we have a contradiction. By similar logic, there cannot be any sinks.

2. For the three parts below, use the following definition: for any digraph $G = (V, E)$ without self-loops and without cycles of length 2 define an undirected graph $G^u = (V, E^u)$ that has the same vertices as $G$ and moreover in $G^u$ we have an edge $v \rightarrow w$ whenever we have the edge $v \rightarrow w$ or the edge $w \rightarrow v$ in $G$.

(a) If $G$ is strongly connected then $G^u$ is connected. Prove or disprove.

Solution:
TRUE. Let $v, w$ be two vertices in $G^u$ (hence in $G$). Since $G$ is strongly connected, there exists a directed walk $v \rightarrow \cdots \rightarrow w$ in $G$. Erasing the direction of the edges in this walk gives a walk $v \rightarrow \cdots \rightarrow w$ in $G^u$. 

(b) If $G$ is a DAG then $G^u$ is acyclic. Prove or disprove.

**Solution:**

FALSE. Counterexample: $G = (\{a, b, c\}, \{(a, b), (a, c), (c, b)\})$ is a DAG but $G^u$ is a cycle of length 3.

(c) Prove that if $G$ is a DAG in which every sink is reachable from every source then $G^u$ is connected.

**Solution:**

Assume $G$ is such a DAG and let’s prove that $G^u$ is connected.

Let $v, w$ be two distinct vertices. We want to show that there is a walk $v \rightarrow \cdots \rightarrow w$ in $G^u$.

**CLAIM** In $G$ there exists a source $s$ such that $v$ is reachable from $s$.

**PROOF OF CLAIM** Consider the set $S$ of all paths in $G$ from some vertex to $v$. There is at least one such path, the path of length 0. By the Well-Ordering Principle at least one of these paths has maximum length among the paths in $S$, we call it a maximal path.

If this path has length 0 then $v$ itself must be a source, otherwise we can extend the length of the path by including one of $v$’s predecessors. So $v$ is reachable from a source (itself).

Suppose the maximal path is not of length 0, let it be $s \rightarrow \cdots \rightarrow v$. Then $s$ must be a source. Indeed, if $s$ has a predecessor $p$ then either $p$ is among the vertices of the maximal path (and then $G$ has a cycle, contradiction) or $p$ is not among the path’s vertices and then the path is not maximal (also a contradiction) because we can extend it with the edge $p \rightarrow s$. So $v$ is reachable from the source $s$. This ends the proof of the claim.

Similarly, we prove the claim.

**CLAIM** In $G$ there exists a sink $t$ such that $t$ is reachable from $w$.

So we have a source $s$ and a sink $t$ such that $s \rightarrow v$ and $w \rightarrow t$. In addition we know that in $G$ every sink is reachable from every source therefore $s \rightarrow t$. Now we erase direction on the walks that give $s \rightarrow v$, $w \rightarrow t$ and $s \rightarrow t$. This gives a walk $v \rightarrow \cdots \rightarrow s \rightarrow \cdots \rightarrow t \rightarrow \cdots \rightarrow w$ in $G^u$. Done.

3. Recall the complete undirected graph on $n$ vertices, $K_n$. Prove that for any $n \geq 4$ it is possible to assign direction to each of the edges of $K_n$ such that the resulting digraph has exactly $n - 2$ strongly connected components.

**Solution:**

**SOLUTION ONE** We proceed by induction on $n$.

**Base case:** $n = 4$. If our graph is

$$\{(a, b, c, 1), ((a, b), (b, c), (c, a), (a, 1), (b, 1), (c, 1))\},$$

we can see that the $n - 2 = 2$ strongly connected components are $\{a, b, c\}$ and $\{1\}$ since the subgraph induced on $\{a, b, c\}$ is a cycle and since $a, b, c$ are not reachable from 1.

**Induction step:** Let $k$ be an arbitrary integer strictly greater than 3, and assume that there is a digraph $G = (V, E)$ with exactly $k - 2$ strongly connected components that is the result of assigning direction to each of the edges of $K_k$ (our induction hypothesis). Now consider the graph $G'$ formed by adding a vertex $v$ to $G$ and directing edges to $v$ from every other vertex. Since no other vertex is reachable from $v$, it forms its own strongly connected component, so $G'$ has $(k + 1) - 2$ strongly connected components, and the induction step holds.
SOLUTION TWO  Let’s name the vertices 1, 2, ..., n, a, b, c (check that there are indeed n vertices).
We have three types of edges in $K_n$: edges between two numbered vertices, edges between two vertices labeled with letters, and edges between one vertex labeled with a number and one with a letter. For the edges with numbers, we orient the edge so that it points from the lower number to the higher number. For the edges with letters, we use the edges $(a, b), (b, c), \text{ and } (c, a)$. For the edges between letters and numbers, we orient the edge so that it points from the letter to the number.

We claim that each vertex 1, 2, ..., n forms a distinct strongly connected component and that \{a, b, c\} forms another. To prove this, we show that no two vertices from different strongly connected components are mutually reachable from each other.

Since every edge that begins in a numbered vertex ends in a numbered vertex, we cannot have a path from a numbered vertex to a non-numbered vertex, so the lettered vertices are not reachable from the numbered vertices and thus cannot be in the same strongly connected component. In addition, since the only edge from any numbered vertex is to a higher-numbered vertex, we cannot have any paths from a higher-numbered vertex to a lower-numbered vertex, and thus the lower-numbered vertices are not reachable from the higher-numbered vertices; therefore no two numbered vertices are in the same strongly connected component, and since they are all separate from the letters, they each constitute their own strongly connected component. Finally, since a, b, and c are part of the cycle $a \rightarrow b \rightarrow c \rightarrow a$, they are in the same strongly connected component.

We have shown that the strongly connected components are \{1\}, \{2\}, ..., \{n - 3\}, \{a, b, c\}. Thus there are exactly $n - 2$, and we are done.

4. Let’s call a slug a DAG $G = (V,E)$ with at least 4 vertices, ($|V| \geq 4$), and such that $G$ has exactly one source $r$ and exactly one sink s.

(a) Draw two different slugs, both with 4 vertices, one of them with 3 edges and the other one with 5 edges.

Solution:

This is a slug with 3 edges:

This is a slug with 5 edges:
(b) Prove that in any slug, for every node \( u \) that is not \( r \) or \( s \), there exists a directed path from \( r \) to \( s \) that passes through \( u \).

**Solution:**
Consider a maximum-length directed path through \( u \), which we call \( v_0, v_1, \ldots, v_k \). We claim that the endpoints of this path are \( r \) and \( s \), i.e. \( v_0 = r \) and \( v_k = s \).

We know that \( in(v_0) = 0 \) or \( in(v_0) > 0 \). If it is the former, then \( v_0 = r \), since there is only one source in the graph. Otherwise, there must be some vertex \( v \) such that \( v \rightarrow v_0 \) is present in the graph. Then \( v \) must already be on the path; otherwise, we could extend it and the path does not have maximum length. Then there is a cycle \( v_0 \rightarrow v \rightarrow v_0 \) (the first part of the cycle is found by following our longest path). Since we know that our graph is a DAG, this is a contradiction. Hence, it must be the case that \( v_0 = r \).

Similarly, we can show that \( v_k = s \) by instead considering the outdegree of \( v_k \). Thus, this path must lead from \( r \) to \( s \) and contain \( u \), as desired.

5. Let’s call Peano-digraph a digraph in which every vertex has outdegree 1.

(a) Prove that any Peano-digraph that is strongly connected is, in fact, a directed cycle.

**Solution:**
Let \( G \) be a strongly connected Peano-digraph. \( G \) cannot have a sink because a sink has outdegree 0. Hence, \( G \) cannot be a DAG, because it would have to have at least one sink, as proved in lecture.

Therefore \( G \) has a directed cycle \( C \) inside it. We will prove by contradiction that all the nodes must be in \( C \) (hence the whole graph coincides with \( C \)).

Suppose, toward a contradiction that \( u \) is a vertex in \( G \) that is not in \( C \). Let \( v \) be a vertex in \( C \). By strong connectivity there is a path \( v \rightarrow u \). Let \( x \rightarrow y \) be the first edge along this path which is not part of the cycle \( C \) (this edge exists because \( v \) is in \( C \) but \( u \) is not in \( C \)). Then \( x \) is in \( C \) but \( y \) is not in \( C \). Since \( x \) already has a successor in \( C \), it must now have two distinct successor which contradicts the fact that the outdegree of \( x \) is 1.

(b) Count the number of different Peano-digraphs whose set of vertices is \([1..n]\), where \( n \) is a positive integer?

**Solution:**
Each vertex is the start of exactly 1 edge. We can choose the end to be any of the \( n \) vertices for each of these edges. Thus, there is a total of \( n^n \) Peano-digraphs.

Another way to count is to notice that there is a 1-1 correspondence between the set of Peano-digraphs with vertices \([1..n]\) and the set of functions with domain \([1..n]\) and codomain \([1..n]\).