1. [10 pts] Jarett loves Halloween because it’s the best time to get free skittles. He knocks on Olivia’s door while trick-or-treating and instead of handing over the candy, she proposes a game. If Jarett decides to decline playing the game, Olivia will give Jarett 15 skittles. Otherwise, he will decide to play the following game:

The game uses two objects: a fair coin and a fair 100-sided die. One side of the coin is labeled with a 1 and the other side is labeled with a 2. Each side of the 100-sided dice roll is labeled with a distinct integer in the interval from 1 to 100, inclusive.

The game consists of two rounds: a coin flip and a 100-sided dice roll. During the first round, he can choose to either flip the coin or to roll the dice. If whichever he chooses results in a 1, he wins 1 skittle and moves on to the next round; if he gets any other number, he must give Olivia 15 skittles and forfeit the game, failing to move on to the next round.

If he makes it to the second round, he must use the other object. For instance, if he rolled the dice in the first round, then he must flip the coin in the second round, and vice versa. In the second round, if he gets a 1, he wins 6160 skittles. If he gets any other number, he loses and must give Olivia 15 skittles.

After hearing the rules, Jarett has three options. He can:

(a) Decide not to play the game.

(b) Play the game and choose to flip a coin for the first round.

(c) Play the game and choose to roll the dice for the first round.
Find the expected value of each of these options. Which option has the highest expected value in skittles?

2. **[12 pts]** While the TAs are out trick-or-treating, they run into a ferocious ghost. They decide to send only the strongest present to fight the ghost and defend their group. To determine who will protect them, each of the $n \geq 2$ TAs will fight every other TA exactly once such that each fight has a winner and a loser. They define an “elite TA” $w$ to be a TA such that, for every other TA $t$, at least one of the following conditions holds:

(1) $w$ beat $t$ when they fought.

(2) $w$ beat some other TA $t'$, and $t'$ beat $t$ when they fought.

Show that there will always be someone who satisfies the above conditions, regardless of the number of TAs and the results of the individual matches.

3. **[14 pts]** For Halloween, the TAs plan on hosting a candy party. For this party, they set up 25 tables in a circle on College Green. There will be 25 TAs who are working at the party, and each of them will stand at a table by themselves. They are allowed to choose any empty table when they show up to the party. Answer the following questions with justification.

   (a) Meri and Ayyah really want to stand at tables next to each other at the party. However, 22 of the 25 TAs who are scheduled to work have already shown up, so there are only 3 empty tables. Assuming that the first 22 TAs arranged themselves at tables uniformly at random (i.e. all arrangements are equally likely), what is the probability that there are 2 empty tables next to each other left for Meri and Ayyah?

   (b) Luckily, they spot 2 empty tables next to each other when they arrive at College Green. Just then, Soham shows up, very late for work! Trying to pretend that he arrived on time, Soham darts in front of them and chooses one of the 3 remaining tables uniformly at random. What is the probability that there will still be two empty tables next to each other after Soham chooses his table?

   (c) Now assume that Soham arrived on time with the first 22 TAs, and Meri and Ayyah have not yet seen which tables the TAs chose. In this scenario, what is the probability of there being 2 empty tables next to each other when Meri and Ayyah arrive at College Green?

4. **[10 pts]** Hari’s Haunted Hillside has $2n$ haunted houses where $n \geq 1$. There are haunted trails connecting pairs of houses, and any pair of houses can have at most one trail between them. Hari noticed that no three houses have all possible trails between them. Specifically, for any
three houses $h_1$, $h_2$, $h_3$, at least one pair of these houses does not have a trail between them. Prove that Hari’s Haunted Hillside has at most $n^2$ trails.

5. [12 pts] Vatsin, embracing his fleeting youth, dresses as an ageless vampire to go trick-or-treating in a neighborhood with $n \geq 3$ houses. Each house can be reached from every other house by a sequence of sidewalk segments, and there is at most one sidewalk segment connecting any two houses. Trick-or-treaters can walk in both directions along any sidewalk segment and each sidewalk segment must connect exactly two distinct houses.

Vatsin is worried about staying out too late and having to walk too far. He especially does not want to waste energy by getting lost and walking in circles, visiting the same houses. Help reassure him by proving that there exists exactly one sequence of sidewalk segments that begins and ends at the same house if and only if there are a total of $n$ sidewalk segments in the neighborhood.

6. [12 pts] Andy has recently turned 20. To celebrate both Andy’s birthday and Halloween, Andy and Kunaal go visit Kunaal’s secret pumpkin farm! At the farm, Kunaal challenges Andy to a special game.

Kunaal has 50 distinct pumpkins set up around his farm. Andy is given one pigeon sticker. In each round, Andy will hide the sticker below one of the 50 pumpkins and bet some amount of money. Kunaal will choose a pumpkin uniformly at random, and if the Andy’s sticker is below that pumpkin, the Andy will be paid 47 times his initial bet. If Andy’s sticker is not below the pumpkin Kunaal chose, Andy will lose his initial bet.

Since it is his 20th birthday, Andy decides that for each round, he is going to place his sticker under the 20th pumpkin and bet $1.

Kunaal then adds a twist to this game. Kunaal bets Andy $25 that after 48 rounds of the game, Andy will have lost more money than he gained. In other words, Andy will pay Kunaal $25 if Andy is behind after 48 rounds; otherwise, Kunaal will pay Andy $25.

(a) Calculate Andy’s expected gain for the 48 rounds only, without Kunaal’s $25 bet. (If he is expected to lose money, his expected gain will be a negative number).

(b) Calculate the probability that Andy is behind at the end of 48 rounds.

(c) Calculate Andy’s expected gain in his bet against Kunaal.

(d) Calculate Andy’s overall expected gain from the game and the bet with Kunaal. Does Kunaal’s bet dissuade Andy from playing?