This assignment is due at the beginning of the class on the due date. Unless all problems carry equal weight, the point value of each problem is shown in [ ]. To receive full credit all your answers should be carefully justified. Please see Piazza for the updated collaboration policy.

Also, please remember to double check that you have submitted the correct version of your homework onto Gradescope by re-downloading it.

1. [10 pts] Here are seven propositions:

\[
\begin{align*}
    &x_1 \lor x_2 \lor \overline{x_7} \lor x_4 \\
    &\overline{x_5} \lor x_6 \lor x_7 \lor x_1 \\
    &x_2 \lor \overline{x_4} \lor x_6 \lor \overline{x_8} \\
    &\overline{x_4} \lor x_5 \lor \overline{x_7} \lor x_6 \\
    &x_3 \lor \overline{x_5} \lor \overline{x_8} \lor x_1 \\
    &x_9 \lor \overline{x_8} \lor x_2 \lor \overline{x_1} \\
    &\overline{x_3} \lor x_9 \lor x_4 \lor x_2
\end{align*}
\]

Note that:

1. Each proposition is the logical or of four distinct terms of the form \( x_i \) or \( \overline{x_i} \).
2. Only one of \( x_i \) or \( \overline{x_i} \) can appear in any particular proposition.
3. No two propositions have the same combination of four terms.

Suppose that we assign true/false values to the variables \( x_1, \ldots, x_9 \) independently and with equal probability.

(a) What is the probability that one particular proposition is true?

(b) What is the expected number of true propositions?

(c) Using only your answer to part (b), is it possible to show that there exists an assignment of the variables such that all of the propositions are true? Why or why not?

2. [10 pts] Nikhil’s new professional networking platform, Linked-N, consists of a number of profiles connected by links, which are two-way connections between two different profiles. Two profiles can share at most one link. Linked-N’s catchy (and surprisingly honest!) marketing slogan is
“Linked-N: Where it is always possible to get from any profile to another by following a sequence of links!”

Given that every profile on Linked-N has an even number of links to other profiles, prove that the slogan still holds even if a sour business deal breaks any one existing link between two profiles.¹

3. [10 pts] Brandon’s new video game company, Lintendo, is selling the new Lintendo Switch game *Pigeon Party* at Store A and Store B. Jonathan goes to Store A with a \( \frac{4}{5} \) chance and Store B with a \( \frac{1}{5} \) chance and buys two copies of the game *Pigeon Party* from the one store he goes to. Every time Jonathan plays a version of the game from store A, he has an independent \( \frac{1}{10} \) chance of losing and every time he plays a version of the game from store B, he has an independent \( \frac{1}{200} \) chance of losing.

Immediately after returning home, Jonathan powers on his Switch and begins playing games on one of the copies of *Pigeon Party*. He keeps playing games, one after the other, until he finally loses a game. Frustrated with losing, Jonathan reaches for the second copy and begins playing games on that copy instead until he loses.

Let \( X \) be a random variable denoting the number of times Jonathan plays on the first copy of *Pigeon Party*, and let \( Y \) be a random variable denoting the number of times Jonathan plays on the second copy. Determine whether or not \( X \) and \( Y \) are independent.

4. [10 pts] Marshall gave each of the \( n \) candies he got from trick-or-treating a score based on how much he likes the candy. No two candies have the same score. Every day he uniformly at random picks one candy from his bag and eats it. Every time Marshall eats a candy \( c_j \), he sheds one disappointed tear for each candy \( c_i \) that he has already eaten which had a higher score than \( c_j \). What is the expected total number of tears Marshall will have shed once he is done eating the \( n \) candies?

5. [10 pts] After Halloween, the \( n \geq 1 \) members of the CIS 160 staff are trading candies. AJ makes the rule that each TA must give candy to at least \( t \) other TA’s. In order to avoid tradebacks, AJ says if Person A gives candy to person B, then Person B can’t give candy to Person A. Prove that AJ will need at least \( 2t \) TA’s in order to orchestrate a successful trading session that meets his requirements.¹

6. [10 pts] To celebrate Día de los Muertos, the 160 staff decided to create a display of 18 sugar skulls in the Bump Space. 10 of the sugar skulls are green, and the other 8 are red. Yunha decides to order the skulls uniformly at random on the table to make the display. In expectation, how many pairs of consecutive sugar skulls are there, such that one skull is green and the other is red?
For example, if the sugar skulls are ordered GRGGGGRRRRRGGGGRRR, we have 5 such pairs.

7. [10 pts] An Eulerian walk in a graph is a walk that traverses each edge exactly once and ends at a vertex other than the one from which it started. Prove that a connected graph with at least two vertices has an Eulerian walk if and only if it has exactly two vertices of odd degree.¹

¹We strongly encourage you to approach this problem without induction (our solution does not use induction). This is not to say it is not possible (correct proofs will receive credit), but induction proofs are not always so simple and it’s important to learn how to construct graph proofs using non-inductive methods as well.