This assignment is due at the beginning of the class on the due date. Unless all problems carry equal weight, the point value of each problem is shown in [ ]. To receive full credit all your answers should be carefully justified. Each solution must be written independently by yourself - Please see Piazza for the updated collaboration policy.

Also, please remember to double check that you have submitted the correct version of your homework onto Gradescope by re-downloading it.

1. [10 pts] Let $G = (V, E)$ be the path graph on $n \geq 2$ vertices, which consists of a single path containing $n$ vertices. When $n$ is even, $G$ contains exactly one maximum size matching of size $\frac{n}{2}$. When $n$ is odd, how many matchings of maximum size exist in $G$, and how many edges are contained in any such matching? Prove your answer. *Hint: try using induction.*

![Figure 1: The path graph on $n = 4$ vertices and its unique maximum matching.](image)

2. [10 pts] Let $Y$ be a random variable such that $Y = \sum_{i=1}^{n} Y_i$, where each $Y_i$ is a random variable.

Prove that if $\mathbb{E}[Y_i Y_j] = \mathbb{E}[Y_i] \mathbb{E}[Y_j]$ for every pair $i, j$ such that $1 \leq i < j \leq n$, then

$$\text{Var}[Y] = \sum_{i=1}^{n} \text{Var}[Y_i]$$

3. [10 pts] Recall that $\chi(G)$ denotes the chromatic number of $G$. Prove that if for any three odd cycles in $G$, at least two share a vertex, then $\chi(G) < 9$. 