This assignment is due at the beginning of the section on the due date. Unless all problems carry equal weight, the point value of each problem is shown in [ ]. To receive full credit all your answers should be carefully justified. Each solution must be written independently by yourself - no collaboration is allowed.

1. [12 pts] Give answers to the following questions.
   (a) List the members of these sets.
       i. \( \{ x \mid x \text{ is prime and less than } 20 \} \)
       ii. \( \{ x \mid x \in \mathbb{N}, x \leq 20 \land (2|x \oplus 3|x) \} \)
       iii. \( \{ x \mid \exists y \in \mathbb{Z}^+ \land \exists z \in \mathbb{Z}^+ \text{ s.t. } x = y(y + z) \land x < 15 \} \)
       iv. \( \{ x \mid x \subseteq \{ a, b, c, d \} \land \{ a, b \} \subseteq x \} \)
   (b) Use the set builder notation to give a description of each of these sets.
       i. \( \{ 1, \frac{2}{4}, \frac{3}{9}, \frac{4}{16}, \frac{5}{25} \} \)
       ii. \( \{ 20, 24, 32, 36 \} \)
       iii. \( \{ 0, 3, 8, 15, 24, 35 \} \)
   (c) What is the cardinality of each of the following sets?
       i. \( \emptyset \)
       ii. \( \{ \{ x \}, y, \{ a, b \} \} \}
       iii. \( \{ a, \{ a \} \} \)
       iv. \( \{ \emptyset \} \)
       v. \( \{ \{ x, \{ y \} \}, \{ a, \{ a \} \}, \{ a, b, c \} \} \)
   (d) Determine whether each of the following is true or false.
       i. \( \emptyset \in \emptyset \)
       ii. \( \emptyset \subseteq \emptyset \)
       iii. \( \{ 1, 2, 3 \} \subseteq \{ 1, 2, \{ 3 \} \} \)
       iv. \( \emptyset \subseteq \{ \{ \emptyset \} \} \)
v. $\emptyset \in \mathcal{P}(\emptyset)$

vi. $\emptyset \subseteq \mathcal{P}(\emptyset)$

(e) Let $A$ be the power set of $\{1, 2, 3\}$. What are $A$ and $|A|$?

(f) Find two sets $A$ and $B$ such that $A \in B$ and $A \subseteq B$.

2. [10 pts] After graduating, former head TA Krishna joins the tech company Elgoog as a product manager. To improve productivity, he imposes a new variable naming convention: all variables in the code must be exactly 10 characters, which can be uppercase letters or lowercase letters.

(a) Given this, how many unique names are possible in the company’s codebase?

(b) How many of these names are composed solely of vowels?

(c) How many of these names are composed solely of consonants?

(d) With productivity increasing, Krishna tries to come up with a new naming convention. The new names require that all names must contain at least one vowel and one consonant in them. Given this, how many unique names are now possible in the company’s codebase?

For this question, consider “y” to be a consonant.

3. [10 pts] Krishna’s next project is to fix the employee username naming schema. He proposes the following:

Each username will consist only of the 26 uppercase letters, the 10 digits, and a space character. Each username is either one word (with no spaces) or two words separated by a space character (note that the space character cannot begin nor end the username). A “word” is a non-empty sequence of letters and numbers. Furthermore, each username must contain exactly 8 distinct characters (including the space character if there are two words). For example, “BIG CAT” is not a valid employee username, but “CISZ 160” and “BIG DUCK” are.

In order to quell the worries that there won’t be enough usernames for all employees, Krishna has asked you to count how many employee usernames are possible using this schema. Can you help him out?

4. [10 pts] As an aspiring designer, Yonah decides to practice by creating a new logo for his website. Yonah decides that the new design should be composed of 6 vertical stripes placed atop of 3 horizontal stripes (as presented in the example picture). The stripes can be colored black, grey, white, red, and gold. A color may be used multiple times, but two stripes of the same color cannot share a long edge.

Yonah starts wondering how many possible designs he could make. Can you help him count the
amount of possible designs there are that fit his description?

As an example, here is one such possible logo:

![Logo Image]

5. [8 pts] Dhruv and Stephanie are in office hours trying to decide where to get dinner. To help them decide, Dhruv writes the numbers 1 through 77 on the whiteboard. They then take turns doing the following: on each turn, they choose two numbers to erase and replace the two numbers with their positive difference (they add back the positive difference just once). They repeat this until a single number remains. If that number is a 0, they will go to Franklin’s Table; otherwise, they will eat at 1920 Commons. Where will Dhruv and Stephanie eat? Explain your answer.

6. [10 pts] Yunha, inspired by the recent World Cup, decides to make soccer balls for her friends. She decides that each ball will have exactly three distinct colors, and for any two colors that she has, those two colors can only appear together on exactly one ball. Now, Yunha is unsure if her plan is always possible and wants your help. If she has an unlimited supply of 100 distinct colors of material, can she use all distinct pairs of colors in her balls? Justify your answer.

7. [10 pts] Angry after losing at fantasy football, Alexandra decides to sabotage the upcoming NFL season, and she curses one of the NFL’s 509 footballs such that anyone who throws it will forget how to play football. Fortunately, the NFL only needs 508 footballs for opening weekend. However, the NFL does not have the time to buy new footballs, so they must resort to finding the cursed one and discarding it. The curse is undetectable until it takes effect 48 hours later, and the first game is only 48 hours away!

Thankfully, 9 brave NFL executives are willing to sacrifice their football abilities in order to help test the footballs. Can you decide how the executives should throw the footballs in order to figure out which football is cursed before the game starts? Justify that your plan always works.

**Hint:** Think Binary