This assignment is due at 9:00AM on the due date. Unless all problems carry equal weight, the point value of each problem is shown in [ ]. To receive full credit all your answers should be carefully justified.

This assignment must be typeset in \LaTeX{} and turned in through Gradescope as a PDF file. Submissions that are not turned in through Gradescope, handwritten homework, or homework that is prepared with other tools (e.g., MS Word), will not be accepted. There will be a 5 point deduction each for incorrectly selecting pages on Gradescope or failing to use the provided template.

Each solution must be written independently by yourself - no collaboration is allowed.

1. [6 pts] Prove or disprove: there do not exist positive integers $a, b, c$ such that

$$a^7 - b^5 = c^4$$

2. [8 pts] The CIS 160 TAs are having a cookout to celebrate the end of summer and the beginning of another semester of CIS 160. Unfortunately, they underestimated everybody’s hunger and have nearly run out of food! To decide who gets to eat the last burger, Kenneth asks the staff to give a proof for $\sqrt{6} + \sqrt{7} < \sqrt{26}$. Tiffany responds instantly with the following:

Squaring both sides of $\sqrt{6} + \sqrt{7} < \sqrt{26}$ gives $13 + 2\sqrt{42} < 26$, which further implies $2\sqrt{42} < 13$. Squaring both sides gives $168 < 169$, which is true.

Help Kenneth verify Tiffany’s solution. If it is valid, give a brief justification why (a couple of lines will suffice). If not, explain why the proof is invalid and provide a correct proof.

3. [22 pts] Answer the following questions. The sets $A, B, C,$ and $D$ are not the same between each subproblem.

(a) Suppose that $A, B,$ and $C$ are sets with $A \cap B \cap C = \emptyset$. Prove or disprove:

$$|A \cup B \cup C| = |A| + |B| + |C|$$

(b) Prove that if $A$ and $B \setminus C$ are disjoint, then $A \cap B \subseteq C$.

(c) Let $A, B, C,$ and $D$ be arbitrary sets. Prove that

$$(A \cap C) \cup (B \cap D) \subseteq (A \cup B) \cap (C \cup D)$$
(d) Let $A$ and $B$ be arbitrary sets. Prove that if $A \neq \emptyset$ and $A \times B = \emptyset$, then $B = \emptyset$.

(e) Let $A$ and $B$ be arbitrary sets. Prove or disprove:

$$\mathcal{P}(A) \cup \mathcal{P}(B) = \mathcal{P}(A \cup B)$$

4. [11 pts] Belinda has brought back $n$ distinct cupcakes from her favorite cupcake shop in New Jersey. She wants to give these to JJ, Ayyah, and Will, who are not convinced that New Jersey has anything worthwhile at all. She wants each cupcake to be tried by at least one person. She may give a whole cupcake to one of her friends, but they are also large enough to split in half to share between two people. However, they are too small to split into three pieces to share between 3 people. Since she wants them to savor the flavors of the Garden State, she will not eat any of the cupcakes herself. Given this, how many ways can Belinda let JJ, Ayyah, and Will sample the cupcakes?

5. [8 pts] After the TAs run out of food, Head TA Hannah makes the executive decision to call up $n$ distinct catering companies. Each catering company has exactly 2 specialty dishes (one spicy dish and one non-spicy dish), each of which Hannah can either buy or refuse to buy. However, Hannah doesn’t want the food selection to be boring, so she doesn’t want to buy two dishes from the same company. Since the TAs have started complaining, she wants to order at least one dish for the cookout. In how many ways can she do this?

6. [15 pts] Andy is cooking steak on the grill and he has 13 different spices that he wants to add to the steak. After carefully examining them for their fragrant qualities and compatibility, he lines them up and labels them from 1 to 13. To best bring out the flavors, a spice $i$ must be added before (not necessarily immediately) at least one of spice $i+1$ or spice $i-1$, if they exist, unless spice $i$ is the last spice added to the steak. For example, spice 1 cannot be added before “spice 0” to satisfy the flavor ordering, since “spice 0” does not exist. Then, spice 1 must be added before spice 2, unless spice 1 is added last. However, spice 2 may be added before spice 1 or spice 3 (or both) to satisfy the flavor ordering, unless spice 2 is added last. In how many orders can Andy add all 13 spices to his steak to bring out the best flavors?