Homework 4 (100 pts)

Posted Thursday, February 21

Due Thursday, February 28, 9AM, uploaded to Gradescope

Logistics  Your homeworks must be typeset in Latex and turned in as a PDF file. Handwritten
homeworks, or homeworks that are prepared with other tools (e.g., MS Word), will not be accepted.
Remember to follow all of the homework guidelines specified on the course website as well as the
collaboration policy which can be found on Canvas.

1. [10 pts] Cecilia goes to Chipotle to order dinner for all of the CIS TAs. Exhausted by the
prospect of ordering so many individual burritos, she gives the cashier all of her money and
allows them to decide what to make. After an hour, they present her with a sack of burritos.
The employees inform her that there are 237 total burritos. Each burrito has at least one kind of
protein: barbacoa, carnitas, or seitan. There are 120 burritos with barbacoa, 110 with carnitas,
and 114 with seitan. The cashier tells Cecilia that 24 of the burritos have barbacoa and seitan,
18 have carnitas and seitan, and 5 burritos have all three proteins.

(a) How many burritos have both barbacoa and carnitas?

(b) 40 of the TAs dislike barbacoa and seitan and will refuse to eat burritos containing either.
Will there be enough burritos that contain only carnitas to feed these TAs?

2. [15 pts] Yonah is the smoothie plug for grading sessions, and is distributing a set $S$ of refreshing,
distinguishable smoothies to the set $T$ of all the TAs. He has graciously procured a nonempty
subset $M$ of special Mango-Pineapple-Strawberry smoothies to satisfy the nonempty subset $P$
of the picky TAs who refuse to have any flavor other than Mango-Pineapple-Strawberry. All
smoothies, including the Mango-Pineapple-Strawberry ones, are distinguishable.

To ensure that each TA gets a smoothie, he wants to use an injective function $f : T \to S$ to
map TAs to smoothies such that for each TA $a \in T$, the smoothie $f(a)$ belongs to TA $a$. Note,
however, that a picky TA can only get a special smoothie, so for every TA $a \in P$, $f(a) \in M$.

For notational convenience, let $|T| = t$, $|S| = s$, $|M| = m$, and $|P| = p$.

(a) What relationships (equalities or inequalities) must exist between $t, s, m,$ and $p$ for Yonah
to be able to assign TAs to smoothies such that everyone is satisfied? Justify your answer.

(b) Given that the relationships you found in part (a) hold, how many different ways can Yonah
assign TAs to smoothies? Your answer should depend upon only the given variables.
3. **[20 pts]** Let $X, Y, Z$ be three finite nonempty sets and let $f : X \to Y$ and $g : Y \to Z$ be two functions. Define a function $h : X \to Z$ such that $h = g \circ f$. Note that the composition $g \circ f$ is defined by $(g \circ f)(x) = g(f(x))$.

Consider the following statements:

- $m$: $f$ is injective
- $n$: $g$ is surjective
- $o$: $h$ is injective
- $p$: $h$ is surjective
- $q$: $h$ is bijective

Prove or disprove each of the following:

(a) $h$ is a function (this is also known as: $h$ is “well-defined”)

(b) $o \implies m$

(c) $p \implies n$

(d) $m \land n \implies q$

(e) $q \implies m \land n$

4. **[15 pts]** Jediah and Ayyah have a major weakness after a fun night out – McDonald’s! Each night they go, Jediah buys 20 Chicken McNuggets and Ayyah buys a milkshake. Each McDonald’s receipt has a positive integer as an order number, and since Jediah and Ayyah stand together in line, their two numbers are always consecutive positive integers.

While they are reminiscing, Jediah comes to the slightly embarrassing realization that he has eaten a whopping $20n$ McNuggets in total, for some $n \in \mathbb{Z}^+$. Ayyah, who has diligently saved all of her and Jediah’s receipts as souvenirs, takes out $n + 1$ of them. Prove that no matter which $n + 1$ receipts Ayyah chooses, two of the receipt numbers are coprime.

5. **[20 pts]** In lieu of taking the CIS 160 exam, Helene, Olivia, Kara, Kenneth, and Kunaal are coordinating a conga line dance routine to present to judges Val and Clayton. So that they can be more easily identified, the five dancers line up so that at least three of them are standing consecutively in alphabetical order by their first names.

Based on the dancers’ arrangement, Val and Clayton will give them a score between 0 and 40, inclusive. Prove that at least two possible dance arrangements would receive the same score.
6. [20 pts] After realizing that the Penn campus is in dire need of a Taco Truck, Tien and Tiffany decide to open the TA Tasty Taco Truck. To craft their tantalizing menu, Tien and Tiffany create a \textit{surjective} function $f : F \rightarrow T$ assigning distinct fillings to tacos, where $F$ is the set of all fillings, and $T$ is the set of all tacos. Note, each filling must exist in exactly 1 taco. In addition, each taco must have exactly $k \geq 1$ fillings.

(a) Give an example of a set of fillings $F$ and a function $g : F \rightarrow T$, where $T = \{1, 2, \ldots, k\}$ is the set of tacos. Show that $g$ satisfies the properties above, namely that $g$ is surjective and exactly $k \geq 1$ taco fillings must be in each taco.

(b) Given that there are $k$ times as many distinct fillings as tacos and there are $n > 0$ tacos, how many different functions can Tiffany and Tien create that satisfy the properties above?