1. [10 pts] After hiding the infinity stones, Mannos now needs 3 of his 40 apprentices to guard the hiding locations. 25 of them have super strength, 30 of them have levitation abilities, 33 of them have super speed, and 35 of them have extremely sharp reasoning skills. Mannos wants each of his 3 guards to have all 4 of these attributes. Prove that there exist at least 3 apprentices such that each apprentice has all 4 of the above attributes.

2. [24 pts] Prove the following using induction.
   (a) \( \forall n \in \mathbb{N}, \ 3^{3n+4} + 2^{n+2} \) is divisible by 5.
   (b) Let \( r, n \in \mathbb{N} \) and let \( r \leq n \). Then
   \[
   \binom{n+1}{r+1} = \sum_{k=r}^{n} \binom{k}{r}
   \]
   (c) Let \( n \in \mathbb{Z}^+ \), and let \( p \) be a prime number. Prove by induction that \( n^p \) can be expressed as the sum of \( n \) and some multiple of \( p \).

3. [10 pts] Give a combinatorial proof to show that for all integers \( n \geq 3 \),
   \[
   3^{n-2} \cdot n \cdot (n-1) = \sum_{k=2}^{n} \binom{n}{k}(k)(k-1)2^{k-2}
   \]

4. [8 pts] After beating Jediah in the demo version of the CIS-themed knock-off game FIFO 2019, AJ decides to give him a consolation gift basket containing exactly \( r \) candies. He goes to Wawa to pick up candies for the basket and sees that they have \( k \) types of candies, with an infinite number of each type of candy \( c_1, c_2, \ldots, c_{k-1} \) but only 1 bar of \( c_k \). In how many ways could AJ create his gift basket from these candies?

5. [8 pts] \( X \) is an \( n \)-subset of \( Y \) if and only if \( X \subseteq Y \) and \( |X| = n \). Let \( S \) be the set of all \( n \)-subsets of the set \( \{1, 2, 3, \ldots, 2n\} \). Assume that \( n \geq 2 \). Prove that \( |S| \) is composite.

6. [10 pts] Amit is planning to organize a Smash Ultimate Tournament this winter but wants to make sure that at least one person will be an ultimate winner. There will be \( n \geq 1 \) participants,
and each participant in the tournament will fight every other participant exactly once in a one-on-one match (each fight will have exactly one winner and exactly one loser – no ties!). Note that victories are not transitive – participant $a$ beating participant $b$ and participant $b$ beating participant $c$ does not imply that $a$ will have beat $c$, as well.

Amit sets the following criteria for a participant to be an ultimate winner: a participant $x$ is an ultimate winner if for all other participants $y$, either $x$ beat $y$ or $x$ beat some third participant $z$ who beat $y$.

Help Amit plan his tournament and prove that at least one of the $n$ participants will be an “ultimate winner”. 