1. [18 pts] Prove the following using induction.
   
   (a) Prove that for all \( n \in \mathbb{Z}^+ \),
   \[
   \sum_{i=1}^{n} i^3 = \left( \sum_{i=1}^{n} i \right)^2
   \]
   
   (b) Prove that for all integers \( n > 1 \),
   \[
   1 + \frac{1}{4} + \frac{1}{9} + \cdots + \frac{1}{n^2} < 2 - \frac{1}{n}
   \]

2. [14 pts] The Mid-Autumn Festival has passed and Cecilia the moon cake chef is giving away her leftover moon cakes. She has 32 identical moon cakes to put into 5 distinguishable gift boxes. She wants four of the boxes to contain an odd number of moon cakes and the last box to contain a number of moon cakes that yields a remainder of 2 when divided by 3, but she doesn’t have a preference as to which box this is. How many ways can Cecilia distribute the moon cakes?

3. [16 pts] Taki the ice cream master is planning the TAs’ office hours ice cream party. He gives each of the TAs an order form, which allows them to select one type of cone, one flavor of ice cream, and one topping (no more, no less). All 38 TAs place their orders, and since they are a variety-loving bunch, they each choose a different type of cone, a different flavor of ice cream, and a different topping.

   (a) Taki wants to assemble the ice cream cones before bringing them to the bump space. Since he is an ice cream master, he knows that ice cream cones must be assembled in an specific order; that is, he can’t scoop ice cream without getting a cone first, and he can’t put on toppings without the ice cream already scooped. However, he doesn’t have to complete any
ice cream cone all at once — for instance, he could get out a cone for one TA, then get out a cone for another TA, then scoop ice cream for the first TA, and so on. How many possible sequences are there for Taki to assemble ice cream cones for all 38 TAs, where the resulting cones are exactly what the TAs ordered, assuming that he must create each ice cream cone in the correct order?

(b) After he gathers exactly as many supplies (cones, ice cream, and toppings) as he needs to fulfill the TAs’ orders, Taki gets distracted thinking about ice cream and loses the order slips. He decides to bring all the supplies to the bump space to hand them out there. However, the ice cream starts to melt, so he doesn’t have any time to recollect orders or allow the TAs to assemble their own ice cream.

He starts handing out supplies to the TAs such that each TA gets one cone, one ice cream scoop, and one topping, but without any regards to their original preferences. Since he is a consummate professional (other than losing everyone’s orders), no TA will receive an ice cream scoop before a cone, or a topping before an ice cream scoop. Again, it is not necessary for him to distribute complete cones all at once, and note that he only has exactly as many supplies as originally gathered. In this scenario, how many possible ways are there for Taki to hand out ice cream cone supplies?

4. [10 pts] Sharon and Vatsin are taking a Buzzfeed quiz to find out what kind of ice cream they are. As they go through the quiz, they are asked to each create a number using mathematical operations and positive integers \( n \) and \( k \), where \( n \geq k \). Sharon decides on \( k\binom{n}{k} \) while Vatsin picks \( n\binom{n-1}{k-1} \). At the end, the quiz tells them that they are both matcha flavored ice cream! Help explain why they got the same result by giving a combinatorial proof for the following identity:

\[
k\binom{n}{k} = n\binom{n-1}{k-1}
\]

where \( k, n \in \mathbb{N}, 1 \leq k \leq n \).

5. [12 pts] Show that the number of \( r \)-combinations of the multiset \( M = \{1 \cdot a_1, \infty \cdot a_2, \ldots, \infty \cdot a_k\} \) is given by

\[
\binom{k + r - 3}{r - 1} + \binom{k + r - 2}{r}
\]

where \( k > 1, r \geq 1 \) and \( k, r \in \mathbb{Z} \). Recall that an \( r \)-combination of the multiset \( M \) is an unordered collection of \( r \) elements from \( M \). Your argument should not involve any kind of algebraic manipulation of the expression.