This assignment is due at the beginning of the class on the due date. Unless all problems carry equal weight, the point value of each problem is shown in [ ]. To receive full credit all your answers should be carefully justified. Each solution must be written independently by yourself - no collaboration is allowed.

Also, please remember to double check that you have submitted the correct version of your homework onto Gradescope by re-downloading it.

1. [23 pts] Hannah is choosing to eat one of \( n \geq 4 \) distinguishable bowls of ramen. The table with the bowls is a little messy and some pairs of bowls \( x, y \) are connected by a noodle which has one end in \( x \) and one end in \( y \). She notices that among any group of four of the bowls of ramen, there will always be at least one bowl in the group that has a noodle from it to each of the other three bowls. Help Hannah prove or disprove that among \( n \) bowls of ramen with the above property, there exists a bowl \( b \) such that for every other bowl \( b' \), \( b \) and \( b' \) have a noodle between them.

2. [23 pts] While waiting for her ramen to cool down, Marn Yee thinks up the following game. She starts with two bowls of ramen in a row: the left one is spicy, the right one is regular. She lets you change the arrangement of bowls in the following manner as many times as you like: you can either insert two consecutive bowls of the same flavor (spicy or regular) OR remove two consecutive bowls of the same flavor. Note that bowls can be inserted into and removed from any consecutive positions that you wish.

Prove that there will never be a point in Marn Yee’s game where there are exactly two bowls in the row, where the left one is regular and the right one is spicy.

(Hint: Try to find a property of the ordering that doesn’t change after any number of turns.)

3. [24 pts] Tired of talking about ramen, Sam instead wants to prove the following inequality for all \( n \in \mathbb{Z}^+ \):

\[
\prod_{i=1}^{n} \frac{2i - 1}{2i} \leq \frac{1}{\sqrt{3n}}
\]

(a) Suppose that we want to prove this statement using induction. Can we let our induction hypothesis simply be the above assertion? Show why this does not work by trying to prove it using induction.

(b) Try proving the claim by instead strengthening the induction hypothesis by changing \( 3n \) to
3n + 1 in the above assertion. In other words, prove:

\[ \prod_{i=1}^{n} \frac{2i - 1}{2i} \leq \frac{1}{\sqrt{3n + 1}} \]

(c) Does proving the new claim in (b) imply what you were trying to prove in part (a)?