1. **[12 pts]** The CIS 160 TAs have obtained a nuclear submarine from who knows where and decide to explore the ocean. Bored on the ride down to the seafloor, Belinda makes a bet with Matt. She rolls a pair of fair distinguishable dice 36 times. If she gets exactly one (1,1), exactly two (2,2)’s, exactly three (3,3)’s, exactly four (4,4)’s, exactly five (5,5)’s, and exactly six (6,6)’s, then Matt will have to pay up big time: his Penn tuition and fees for next year ($57,770). If Belinda loses, she will have to give Matt the pretty seashell she found ($5 value).

What’s Belinda's expected payoff from the bet? No need to simplify or evaluate your answer.

2. **[20 pts]** Sid the scuba diver is captured by Jediah the shark and forced to participate in an underwater Monty Hall game show (see lecture 14) to win his freedom. Behind one door lies his freedom; the other two doors mean certain death in a shark pit.

Instead of using what he has learned in CIS 160, Sid decides to listen to his magic conch. With probability $p$, the conch will tell Sid to switch; otherwise, he will stay with the first door he chooses. You may assume that Sid chooses doors uniformly at random and Jediah acts as Monty from the Monty Hall game.

(a) Compute, in terms of $p$, the probability that Sid chooses the door that secures his freedom.

(b) What is the smallest value of $p$ that still gives Sid a 50% or better chance of surviving?

(c) Right before Sid chooses his first door, a sneaky crab tampers with the magic conch and sets $p$ to be $1/5$. After the crab tampers with the conch, it is spotted by Jediah and flees immediately; therefore, it has no idea what Sid chooses. All it knows is that Sid does not survive. Given this, what is the probability that Sid did not switch doors?

3. **[15 pts]** Let $A, B, C$ be three events in the same probability space. We say that $A$ is condi-
tionally independent of $B$ given $C$, written $A \perp B \mid C$, when

$$\Pr[A \cap B \mid C] = \Pr[A \mid C] \cdot \Pr[B \mid C]$$

Soham’s laptop dies in the Fish Bowl just as he is about to spam his $160^{th}$ Leetcode question. Since he has nothing to do until Ahmed swims by with a spare laptop charger, he takes out his wallet (which is otherwise empty) and decides to flip 3 fair coins for fun. He picks three events out of the sample space $\Omega$ defined by the 3 coin tosses and calls them $A$, $B$, and $C$. He also wants $A \perp B$ and $A \not\perp B \mid C$.

Give an example of events $A$, $B$, and $C$ that satisfy the above conditions and justify.

4. [15 pts] One day, while bravely exploring the ruins of the Titanic, mermaid Zach stumbles upon a safe that contains the lost treasures of the Ocean. The safe is protected by a password that is the product of two 2-digit numbers $x$ and $y$, such that each digit of the two numbers is a number from $[1..4]$ chosen uniformly at random. For example, if the four random digits are $1, 2, 3, 4$, then $x = 12$ and $y = 34$. What’s the expected value of $x \cdot y$?

5. [20 pts] Meri the magnificent diver is going on a dive one day with some friendly dolphins. There are $n$ distinct dolphins in the area that Meri is diving in. Every second, one of the $n$ dolphins uniformly at random comes by to say hello, high-fives Meri, and then swims away. Note the same dolphin may come multiple times (it’s just very friendly!). However, by the $k^{th}$ dolphin ($k \in \mathbb{Z}^+$), Meri needs to swim back up for air and she stops high fiving dolphins. At this point (after $k$ seconds), what is the expected number of dolphins that Meri didn’t high five?

6. [18 pts] While exploring deep sea trenches, Ying realizes he has discovered a new species of aquatic squirrels and decides to capture. His submarine has a robotic arm that can safely catch the squirrels underwater, but due to poorly written software has a $1/2$ chance of failing to catch anything (independent of any previous attempts). Throughout the day, Ying tries to catch squirrels with the robotic arm $2n$ times. Given an integer $k$, $1 \leq k \leq n$, what is the probability that Ying manages to catch a squirrel in each of his first $k$ attempts but fails to catch any squirrels on his last $k$ attempts, given that over the course of the day the robotic arm succeeded the same number of times it failed?