Homework 8 (100 pts)

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Due Thursday, April 11, 9AM, uploaded to Gradescope

Logistics Your homeworks must be typeset in \LaTeX and turned in as a PDF file. Handwritten homeworks, or homeworks that are prepared with other tools (e.g., MS Word), will not be accepted. Remember to follow all of the homework guidelines specified on the course website as well as the collaboration policy which can be found on Canvas.

1. [20 pts] The CIS 160 staff have been captured by the evil scientist Dr. ClayShrinkenstein. The nefarious Dr. ClayShrinkenstein has decided to test his newest nefarious technology, the shrink ray, the CIS 160 staff to shrink them down to the size of ants! Let $Q$ be the event that Quincy gets shrunk, and $B$ be the event that Brian gets shrunk; suppose these are random events over some probability space $(\Omega, \Pr)$. Recall that we can define an indicator random variable $I_S$ that is 1 if event $S$ occurs and 0 otherwise.

(a) Prove that $I_Q I_B = I_{Q \cap B}$ by showing that they are equal for any input. Note that $I_Q I_B(\omega) = I_Q(\omega) \cdot I_B(\omega)$.

(b) Suppose that $\mathbb{E}[I_Q I_B] = \mathbb{E}[I_Q] \times \mathbb{E}[I_B]$. Prove that $I_Q \perp I_B$.

2. [15 pts] The tiny CIS 160 TAs figure their best chance at survival is to band together into the little colony of Inducteeny. Reconnaissance officer Jediah discovers that some of the more aggressive squirrels on campus are about to launch an invasion. To defend the colony, he needs to assign 9 TAs to 17 potential battle posts, so he assigns each TA to one of the 17 battle posts uniformly at random. Note that there may be more than one TA at each post. Underling TA Soham tells him that he has made a grave mistake—that every TA might be assigned to the same post, leaving the rest unattended—but Jediah insists his method will, in expectation, leave enough posts covered.

Let $X$ be the random variable representing the number of occupied posts. Jediah tasks you with finding evidence to support his argument.

(a) Compute $\mathbb{E}[X]$.

(b) Compute $\text{Var}[X]$.

3. [15 pts] To defend the Inducteeny colony against future invasions, the TAs build a shelter network under the E-Quad. Each pair of shelter rooms is connected by at most one tunnel.
They want to designate a room that is connected to at least 10 other rooms as their meeting room. There are 27 rooms in total, and the builder of each room tells the chief engineer Stephanie how many tunnels are connected to that room. After adding the numbers up, Stephanie noticed that the sum is an integer that is at least 243. She hence declares “we must have a room that can be our meeting room!” Prove that Stephanie is correct, which is to say there must exist a room that is connected to at least 10 other rooms.

4. [15 pts] To help feed their growing Inducteeny colony, $n \geq 6$ tiny TAs travel to the barbaric wilderness, Huntzmen Hell, to find berries. To make sure they don’t get stepped on by the Evil Belinda Cthu-Liu, they decide to travel in exactly $n - 3$ berry-picking parties into the wild land. Brandon creates an equivalence relation on the set of TAs such that the TAs in each berry-picking party form an equivalence class. How many such relations can Brandon construct?

5. [15 pts] Mini Gautam and Zach take a break from their studies at 5 AM one morning to scavenge for energy drinks in the McClelland dumpsters (only a few drops will suffice, given their tiny bodies). While searching for their caffeine fix, Zach and Gautam stumble upon a sophisticated rat community. Each rat in this community has its own home, and each of the homes can have at most one underground tunnel to each other home. It is possible to get from any home to any other home through some series of underground tunnels. Let $k$ be the maximum number of tunnels that you can pass through in a row without visiting the same home twice. Prove that any two paths that pass through exactly $k$ tunnels without visiting the same home twice must pass through a common home.

6. [20 pts] Some of the more entrepreneurial tiny TAs (the [M&]TAs) establish a tiny conglomerate that consists of 8 tiny corporations. Between every group of 3 tiny corporations, at least two have a tiny mutual business agreement between them. You may assume that at most one tiny mutual business agreement can exist between any two tiny corporations, and that no tiny corporation has a tiny mutual business agreement with itself.

(a) Draw a conglomerate with 8 corporations and 12 mutual business agreements satisfying these conditions. For your drawing, it is sufficient to represent corporations as circles and mutual business agreements as lines going between circles.

(b) Show that it is impossible for such a conglomerate to exist with fewer than 12 mutual business agreements between them.

*Hint: What happens when each corporation has at least three mutual business agreements?*