Homework 9 (100 pts)

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Due Thursday, April 18, 9AM, uploaded to Gradescope

Logistics  Your homeworks must be typeset in \LaTeX and turned in as a PDF file. Handwritten homeworks, or homeworks that are prepared with other tools (e.g., MS Word), will not be accepted. Remember to follow all of the homework guidelines specified on the course website as well as the collaboration policy which can be found on Canvas.

After experimenting with new artistic styles over the past semester, the CIS 160 TAs are pioneering a new avant-garde movement that they call Discrete Art. They are proud to present the fruit of their labor in the Discrete Art exhibition.

The landmark works of Discrete Art presented at the exhibition seek to investigate the nature of space, connection, and direction. Each piece offers a novel perspective that will subvert the viewer’s preconceptions. While a sampling of the works are displayed here, the artists have requested that the viewers attempt to reconstruct the art as their own creative outlet before the final reveal.

1. [18 pts] In order to gain admission to the Discrete Art exhibit, please state whether the following claims are True or False. If True, prove the claim. If False, disprove the claim.

   (a) Given binomial random variables $A$ and $B$ with the same parameter $p \in \mathbb{R}$, where $0 < p < 1$, then $A + B$ is also a binomial random variable.

   (b) In any graph $G$, every vertex with odd degree has a path to an odd number of other vertices with odd degree.

   (c) There exists a tree $T$ with 5 leaves, an equal number of degree-2 and degree-3 vertices, a maximum degree of 3, and $n \geq 15$ vertices in total, for some $n$.

2. [17 pts] Renowned art critic Kara has been staring at a particularly abstract work of art – Picassoham’s piece, $T$ (2014) – for several hours. Suddenly, Kara notices that $T$ is a graph with $n \in \mathbb{Z}^+$ blotches of paint and $n - 1$ lines, where each line connects two paint blotches together, and that $T$ is acyclic. She declares that the artwork depicts a tree!

   (a) Using the corollary on slide 18 of lecture 20, help Kara prove that $T$ is a tree.

   (b) Without using (or re-proving) the above corollary, again show that $T$ is a tree.
3. [15 pts] Jedalı has been thinking a lot lately about duality – mind and body, food and sleep, theoretical computer science and even more theoretical computer science – so his piece is a diptych (it has two parts). For the first part, titled $G$, he splatters $n$ dots of paint onto the canvas, and joins some pairs of distinct dots with at most one line. For the second part, titled $\overline{G}$, he splatters the same $n$ dots of paint onto the other side of the canvas, but he joins two dots with a line in $\overline{G}$ if and only if they were not joined in $G$.

To his great dismay, most viewers cannot tell $G$ and $\overline{G}$ apart; aspiring critic Kenneth describes the pieces as “isomorphic” in his review. In your own art review, prove that if $G$ and $\overline{G}$ are isomorphic, then $n$ cannot be oddly even (that is, twice an odd number).

![Figure 1: Example of Jedali’s piece, The Persistence of Parity (1931), if $n = 4$. Oil on canvas.]

4. [15 pts] Rembrandon proposes a new painting for the exhibit. He first draws $n - 1$ blotches and labels them $1, 2, \ldots, n - 1$. He adds a stroke connecting every blotch $i$ with blotch $i + 1$ where $1 \leq i \leq n - 2$, and a stroke connecting blotch $n - 1$ with blotch $1$. After the review panel rejects his work, he attempts to make it more interesting and adds a new blotch, labeled $0$. He also adds $n - 1$ strokes connecting the new blotch with all of the other blotches. During his second visit to the review panel, they inform Rembrandon that his work will be selected if he answers the following questions correctly:

- (a) How many cycles of length 3 are there?
- (b) How many paths of length 2 are there?
- (c) How many cycles are there in total?

![Figure 2: Rembrandon’s piece, The Node Watch (1642). Oil on canvas.]

5. **[15 pts]** The *Discrete Art* exhibit consists of a number of rooms, some of which are joined directly by at most one hallway. Obeying the principle of *form follows function*, Warholiver’s piece is actually a map of the exhibit, where he places a pin on the canvas for each room, and ties a yarn between two pins if the corresponding rooms are joined by a hallway.

Warholiver defines \( d(u, v) \) to be the fewest number of hallways one must walk through in order to pass from room \( u \) to room \( v \) without passing through the same room twice. Suppose \( v_0 - v_1 - \ldots - v_n \) is a list of distinct rooms connected by hallways such that \( d(v_0, v_n) = n \geq 3 \). Prove that \( d(v_0, v_i) = i \), for all \( 0 \leq i \leq n - 3 \).

6. **[20 pts]** Vatsin van Gogh, in a sudden epiphany, realizes that the most meaningful art is the friends he’s made along the way. He decides to draw out all the friendships in CIS 160, painting each student in the class on a massive canvas and connecting them with a line if they’re friends (no student is friends with themselves, and at most one line is ever drawn between two given students). While planning out the painting, Vatsin realizes that he really wants there to be at least one cycle of even length among the friends. What is the minimum value of an integer \( k \) such that if every student has at least \( k \) friends, there must be at least one cycle of even length?

7. **[0 pts]** The birds are chirping, the sun is shining, and it’s a beautiful day to take a walk down Locust. Everyone is having a picnic in Penn Park, although a few people are missing. Stephanie and Brandon were left behind on Jediahamas, but are having such a good time that they don’t mind. Ayyah has become a renowned astronaut, and is about to embark upon her 160\(^{th}\) mission. Sid successfully escaped from the shark, and decided to keep exploring the seafloor. Zach and Gautam couldn’t un-shrink themselves, but have been hired by an entomology lab to search for rare insects. And Jediah has quit his day job as a teaching assistant to manage a *Discrete Art* gallery full-time.

As a newly minted master of the *Mathematical Foundations of Computer Science*, your potential is **unlimited**. So what would you like to do next?

(a) Join the *Discrete Art* gallery to become a legendary artist.

(b) Shrink down to the size of an ant and hunt for rare insects.

(c) Explore the ocean depths to search for Atlantis.

(d) Embark on a mission of your own to outer space.

(e) Take a vacation to Jediahamas.