Introduction to Logic

A proposition is a statement that is either true or false. For example, “2 + 2 = 4” and “Donald Knuth is a faculty at Rutgers-Camden” are propositions, whereas “What time is it?” , \( x^2 < x + 40 \) are not propositions.

We can construct compound propositions from simpler propositions by using some of the following connectives. Let \( p \) and \( q \) be arbitrary propositions.

Negation: \( \bar{p} \) (read as “not \( p \)” ) is the proposition that is true when \( p \) is false and vice-versa.

Conjunction: \( p \land q \) (read as “\( p \) and \( q \)” ) is the proposition that is true when both \( p \) and \( q \) are true.

Disjunction: \( p \lor q \) (read as “\( p \) or \( q \)” ) is the proposition that is true when at least one of \( p \) or \( q \) is true.

Exclusive Or: \( p \oplus q \) (read as “\( p \) exclusive-or \( q \)” ) is the proposition that is true when exactly one of \( p \) and \( q \) is true.

Implication: \( p \rightarrow q \) (read as “\( p \) implies \( q \)” ) is the proposition that is false when \( p \) is true and \( q \) is false and is true otherwise.

The implication \( q \rightarrow p \) is called the converse of the implication \( p \rightarrow q \). The implication \( \neg p \rightarrow \neg q \) is called the inverse of \( p \rightarrow q \). The implication \( \neg q \rightarrow \neg p \) is the contrapositive of \( p \rightarrow q \). \textit{p only if q} means “if not \( q \) then not \( p \)”, or equivalently if \( p \) then \( q \).

Biconditional: \( p \leftrightarrow q \) (read as “\( p \) if, and only if, \( q \)” ) is the proposition that is true if \( p \) and \( q \) have the same truth values and is false otherwise. “If and only if” is often abbreviated as iff.

The following truth table makes the above definitions precise.

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<tr>
<th>( p )</th>
<th>( q )</th>
<th>( \neg p )</th>
<th>( p \land q )</th>
<th>( p \lor q )</th>
<th>( p \oplus q )</th>
<th>( p \rightarrow q )</th>
<th>( q \rightarrow p )</th>
<th>( p \leftrightarrow q )</th>
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Necessary and Sufficient Conditions: For propositions \( p \) and \( q \),
p is a sufficient condition for q means that \( p \rightarrow q \).

p is a necessary condition for q means that \( \neg p \rightarrow \neg q \), or equivalently \( q \rightarrow p \).

Why is \( p \land q \) not the correct answer?

Thus p is a necessary and sufficient condition for q means “p iff q”.

### Logical Equivalence

Two compound propositions are logically equivalent if they always have the same truth value. Two statement p and q can be proved to be logically equivalent either with the aid of truth tables or using a sequence of previously derived logically equivalent statements.

**Example.** Show that \( p \rightarrow q \equiv \neg p \lor q \equiv \neg q \rightarrow \neg p \).

**Solution.** The truth table below proves the above equivalence.

\[
\begin{array}{cccccccc}
p & q & \neg p & \neg q & p \rightarrow q & \neg p \lor q & \neg q \rightarrow \neg p \\
T & T & F & F & T & T & T \\
T & F & F & T & F & F & F \\
F & T & T & F & T & T & T \\
F & F & T & T & T & T & T \\
\end{array}
\]

**Example.** Show that \( p \equiv \neg p \rightarrow C \) and \( p \rightarrow q \equiv (p \land \neg q) \rightarrow C \).

\[
\begin{array}{cccccccc}
p & q & \neg p & \neg q & p \rightarrow q & p \land \neg q & \neg p \rightarrow C & (p \land \neg q) \rightarrow C \\
T & T & F & F & T & T & F & T \\
T & F & F & F & T & F & F & T \\
F & T & T & F & T & T & T & F \\
F & F & T & T & T & T & F & T \\
\end{array}
\]

The above equivalence forms the basis of proofs by contradiction.

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### The logic of Quantified Statements

Consider the statement \( x < 15 \). We can denote such a statement by \( P(x) \), where \( P \) denotes the predicate “is less than 15” and \( x \) is the variable. This statement \( P(x) \) becomes a proposition when \( x \) is assigned a value. In the above example, \( P(8) \) is true while \( P(18) \) is false.

Another way to convert the statement \( P(x) \) into a proposition is through quantification. The two types of quantification that we will study are universal quantification and existential quantification. Using universal quantifier \( \forall \) (“for all”) alongside \( P(x) \) means that the statement \( P(x) \) is true for all elements in the domain of \( x \). Thus the proposition \( \forall x \in D, P(x) \) is true when \( P(x) \) is true for all \( x \in D \) and is false if there is an element
$x' \in D$ for which $P(x')$ is false. Using existential quantifier $\exists$ (“there exists”) alongside $P(x)$ means that there exists an element in the domain of $x$ for which $P(x)$ is true. Thus the proposition $\exists x \in D, P(x)$ is true if there is an $x' \in D$ for which $P(x')$ is true and is false if $P(x)$ is false for all $x \in D$.

Examples of propositions using quantifiers are as follows.

1. $\forall x \in \mathbb{Z}, x^3 + 1$ is composite.
2. $\forall x \in \mathbb{Z}, x$ is even $\rightarrow$ $x + 1$ is odd.
3. $\exists x \in \mathbb{N}, x^2 \neq x$.
4. $\exists x \in \mathbb{Z}, 2|x$ and $2|x + 1$.
5. $\forall x \in \mathbb{Z} \exists y \in \mathbb{Z}, x + y = 0$.
6. $\exists x \in \mathbb{Z} \forall y \in \mathbb{Z}, x > y$.

Sometimes it helps (in proofs) to consider the negation of a proposition. Verify the following equivalence.

$$
\neg (\forall x \in D, P(x)) \equiv \exists x \in D, \neg P(x)
$$

$$
\neg (\exists x \in D, P(x)) \equiv \forall x \in D, \neg P(x)
$$

Proofs

We will illustrate some proof techniques by proving some properties about numbers. Before we do that let’s go through some basic definitions given below.

An integer $n$ is even iff $n = 2k$ for some integer $k$. An integer is odd iff $n = 2k + 1$ for some integer $k$. Symbolically,

$$
n \text{ is even } \iff \exists \text{ an integer } k \text{ s.t. } n = 2k
$$

$$
n \text{ is odd } \iff \exists \text{ an integer } k \text{ s.t. } n = 2k + 1
$$

An integer $n$ is prime iff $n > 1$ and for all positive integers $r$ and $s$, if $n = r \cdot s$, then $r = 1$ or $s = 1$. Otherwise $n$ is composite.

Given any real number $x$, the floor of $x$, denoted by $\lfloor x \rfloor$, is defined as follows

$$
\lfloor x \rfloor = n \iff n \leq x < n + 1, \text{ where } n \text{ is an integer}
$$

Given any real number $x$, the ceiling of $x$, denoted by $\lceil x \rceil$, is defined as follows

$$
\lceil x \rceil = n \iff n - 1 < x \leq n, \text{ where } n \text{ is an integer}
$$

A real number is rational iff it can be expressed as a ratio of two integers with a non-zero denominator. A real number that is not rational is irrational. More formally,

$$
r \text{ is rational } \iff \exists \text{ integers } a \text{ and } b \text{ such that } r = a/b \text{ and } b \neq 0.
$$
Example. Prove the following: If the sum of two integers is even then so is their difference.

Solution. Let $m$ and $n$ be particular but arbitrarily chosen integers such that $m + n$ is even. By definition of even, we have $m + n = 2k$, for some integer $k$. Then

$$m = 2k - n$$

Now $m - n$ can be written as follows.

$$m - n = 2k - n - n = 2(k - n)$$

Since $k$ and $n$ are integers, $k - n$ is an integer, $2(k - n)$ is even and hence $m - n$ is even.

Example. Prove that, for all integers $n$, if $n$ is odd then $n^2 + n + 1$ is odd.

Solution. Since $n$ is odd $n = 2k + 1$ for some integer $k$. Then,

$$n^2 + n + 1 = (2k + 1)^2 + 2k + 1 + 1$$

$$= 4k^2 + 4k + 1 + 2k + 2$$

$$= 4k^2 + 6k + 2 + 1$$

$$= 2(2k^2 + 3k + 1) + 1$$

Since $k$ is an integer, $p = 2k^2 + 3k + 1$ is an integer and $n^2 + n + 1$ is odd, since $n^2 + n + 1 = 2p + 1$ where $p$ is an integer.

Example. Let $x$ be an integer. If $x > 1$, then $x^3 + 1$ is composite.

Solution. Let $x$ be an arbitrary but specific integer such that $x > 1$. We can rewrite $x^3 + 1$ as $(x + 1)(x^2 - x + 1)$. Note that since $x$ is an integer both $(x + 1)$ and $(x^2 - x + 1)$ are integers. Hence $(x + 1)|x^3 + 1$ and $(x^2 - x + 1)|x^3 + 1$. We now need to show that $x + 1 > 1$ and $x^2 - x + 1 > 1$. Since $x > 1$, clearly, $x + 1 > 1$. $x^2 - x + 1 > 1$ by the following reasoning.

$$x > 1$$

$$x^2 > x$$ (Multiplying both sides by $x$.)

$$x^2 - x > 0$$ (Subtracting both sides by $x$.)

$$x^2 - x + 1 > 1$$ (Adding 1 to both sides.)

We can also argue that $x^2 - x + 1 > 1$ by showing that $x + 1 < x^3 + 1$. Since $x > 1$ we have $x^2 > x$ and hence $x^2 > 1$. Multiplying both sides by $x$ again we get $x^3 > x$. This means that $x + 1 < x^3 + 1$ and since $(x + 1)|x^3 + 1$, we conclude that $x^3 + 1$ is composite.

Note: One student asked the question that why can’t we write $x^3 + 1$ as $x^3(1 + \frac{1}{x^3})$. The reason is that for an integer $x > 1$, $(1 + \frac{1}{x^3})$ is not an integer and the proof breaks down.