Example. Prove that the sum of the first $n$ positive odd numbers is $n^2$.

Solution. We want to prove that $\forall$ positive integers $n$, $P(n)$ where $P(n)$ is the following property.

$$\sum_{i=0}^{n-1} 2i + 1 = n^2$$

Base Case: We want to show that $P(1)$ is true. This is clearly true as

$$\sum_{i=0}^{0} 2i + 1 = 1 = 1^2$$

Induction Hypothesis: Assume $P(k)$ is true for some $k \geq 1$.

Induction Step: We want to show that $P(k+1)$ is true, i.e., we want to show that

$$\sum_{i=0}^{k} 2i + 1 = (k+1)^2$$

We can do this as follows.

$$\sum_{i=0}^{k} 2i + 1 = \sum_{i=0}^{k-1} 2i + 1 + 2k + 1$$

$$= k^2 + 2k + 1 \quad \text{(using induction hypothesis)}$$

$$= (k + 1)^2$$

Example. Show that for all integers $n \geq 0$, if $r \neq 1$,

$$\sum_{i=0}^{n} ar^i = \frac{a(r^{n+1} - 1)}{r - 1}$$
Solution. Let $r$ be any real number that is not equal to 1. We want to prove that $\forall$ integers $n$, $P(n)$, where $P(n)$ is given by

$$\sum_{i=0}^{n} ar^i = \frac{a(r^{n+1} - 1)}{r - 1}$$

**Base Case:** We want to show that $P(0)$ is true.

$$\sum_{i=0}^{0} ar^i = a = \frac{a(r - 1)}{r - 1}$$

**Induction Hypothesis:** Assume that $P(k)$ is true for some $k \geq 0$.

**Induction Step:** We want to show that $P(k + 1)$ is true, i.e., we want to prove that

$$\sum_{i=0}^{k+1} ar^i = \frac{a(r^{k+2} - 1)}{r - 1}$$

We can do this as follows.

L.H.S. $= \sum_{i=0}^{k+1} ar^i$

$= \sum_{i=0}^{k} ar^i + ar^{k+1}$

$= \frac{ar^{k+1} - a}{r - 1} + ar^{k+1}$

$= \frac{a(r^{k+1} - 1)}{r - 1} + \frac{ar^{k+1}(r - 1)}{r - 1}$

$= \frac{a}{r - 1} (r^{k+1}(1 + r - 1) - 1)$

$= \frac{a}{r - 1} (r^{k+2} - 1)$

$= \frac{a(r^{k+2} - 1)}{r - 1}$

**Example.** Prove that $\forall$ non-negative integers $n$,

$$\sum_{i=0}^{n} 2^i = 2^{n+1} - 1$$

**Solution.** By setting $a = 1$, $r = 2$ in the result of the previous problem, the claim follows.
Example. Prove that $\forall$ non-negative integers $n$, $2^{2n} - 1$ is a multiple of 3.

Solution. We want to prove that $\forall$ non-negative integers $n$, $P(n)$, where $P(n)$ is

$$2^{2n} - 1 = 3k,$$

for some non-negative integer $k$

Base Step: $P(0)$ is true as shown below.

$$2^0 - 1 = 0 = 3 \cdot 0.$$

Induction Hypothesis: Assume that $P(x)$ is true for some $x \geq 0$, i.e., $2^{2x} - 1 = 3 \cdot k'$, for some $k' \geq 0$.

Induction Step: We want to prove that $P(x + 1)$ is true, i.e., we want to show that

$$2^{2(x+1)} - 1 = 3l,$$

for some non-negative integer $l$.

We can show this as follows.

\[
\text{L.H.S.} \quad = \quad 2^{2(x+1)} - 1 \\
\quad = \quad 2^{2x+2} - 1 \\
\quad = \quad 2^{2x} \cdot 2^2 - 1 \\
\quad = \quad 2^{2x} \cdot 4 - 1 \\
\quad = \quad 2^{2x} \cdot (3 + 1) - 1 \\
\quad = \quad 3 \cdot 2^{2x} + 2^{2x} - 1 \\
\quad = \quad 3 \cdot 2^{2x} + 3 \cdot k' \quad \text{(using induction hypothesis)} \\
\quad = \quad 3(2^{2x} + k') \\
\quad = \quad 3l, \quad \text{where } l = 2^{2x} + k'
\]

Since $x$ and $k'$ are integers $l$ is also an integer. Hence, $P(x + 1)$ is true.

Example. Prove that $\forall n \in \mathbb{N}, n > 1 \rightarrow n! < n^n$.

Solution. Below is a simple direct proof for this inequality.

$$n! = 1 \times 2 \times 3 \times \cdots \times n$$

$$< n \times n \times n \times \cdots \times n$$

$$= n^n$$

We now give a proof using induction. Let $P(n)$ denote the following property.

$$n! < n^n$$

Induction Hypothesis: Assume that $P(k)$ is true for some $k > 1$.

Base Case: We want to prove $P(2)$. $P(2)$ is the proposition that $2! < 2^2$, or $2 < 4$, which
We want to prove that the claim is true when \( n \).

**Induction Hypothesis:** We will prove the claim using induction on \( n \).

Base Case: \( n = 1 \). When \( S = \{x_1, x_2, \ldots, x_n\} \), we want to show that if \( S \) does not contain \( x_{k+1} \), and \( S_2 \subset S \) contains subsets of \( S \) that contains \( x_{k+1} \). Thus we have

\[
|\mathcal{P}(S)| = |S_1| + |S_2|
\]  

(1)

Note that \( S_1 \) contains all subsets of \( \mathcal{P}(S') \). By the induction hypothesis, we have \( |S_1| = |\mathcal{P}(S')| = 2^k \). We will now compute \( |S_2| \). Observe that each set in \( S_2 \) is of the form \( \{x_{k+1}\} \cup X \), where \( X \) is a subset of \( S' \). By induction hypothesis, we know that there are \( 2^k \) subsets of \( S' \) and hence \( |S_2| = 2^k \). Plugging in the values for \( |S_1| \) and \( |S_2| \) in (1), we get

\[
|\mathcal{P}(S)| = 2^k + 2^k = 2^{k+1}
\]

**Example** Let \( A_1, A_2, \ldots, A_n \) be sets (where \( n \geq 2 \)). Suppose for any two sets \( A_i \) and \( A_j \) either \( A_i \subseteq A_j \) or \( A_j \subseteq A_i \). Prove by induction that one of these \( n \) sets is a subset of all of them.

**Solution.** We will prove the claim using induction on \( n \).

**Induction Hypothesis:** Assume that the claim is true when \( n = k \), for some \( k \geq 2 \). In other words, assume that if we have sets \( A_1, A_2, \ldots, A_k \), where for any two sets \( A_i \) and \( A_j \), either \( A_i \subseteq A_j \) or \( A_j \subseteq A_i \), then one of the \( k \) sets is a subset of all of the \( k \) sets.

**Base Case:** \( n = 2 \). We have two sets \( A_1, A_2 \) and we know that \( A_1 \subseteq A_2 \) or \( A_2 \subseteq A_1 \). Without loss of generality assume that \( A_1 \subseteq A_2 \). Then \( A_1 \) is a subset of \( A_1 \) and is also a
subset of $A_2$, so the claim holds when $n = 2$.

Induction Step: We want to prove the claim when $n = k + 1$. That is, we are given a set $S = \{A_1, A_2, \ldots, A_{k+1}\}$ of with the property that for every pair of sets $A_i \in S$ and $A_j \in S$, either $A_i \subseteq A_j$ or $A_j \subseteq A_i$. We want to show that there is a set in $S$ that is a subset of all $k+1$ sets in $S$. Let $S' = S \setminus \{A_{k+1}\}$. By induction hypothesis, there is a set $A_p \in S'$ that is a subset of all sets in $S'$. We now consider the following two cases.

Case 1: $A_p \subseteq A_{k+1}$. Then it follows that $A_p$ is a subset of all sets in $S$.

Case 2: $A_{k+1} \subseteq A_p$. Since $A_p$ is a subset of all sets in $S'$ and $A_{k+1} \subseteq A_p$, it follows that $A_{k+1}$ is a subset of all sets in $S$.

Example. For all $n \geq 1$, prove that $n$ lines separate the plane into $(n^2 + n + 2)/2$ regions. Assume that no two of these lines are parallel and no three pass through a common point.

Solution. Let $P(n)$ be the property that $n$ lines, such that no two of them are parallel and no three of them pass through a common point, separate the plane into $(n^2 + n + 2)/2$ regions. We will prove the claim by induction on $n$.

Induction Hypothesis: Assume that $P(k)$ is true for some $k > 0$.

Base Case: $P(1)$ is true since one line divides the plane into 2 regions which is also given by $(1^2 + 1 + 2)/2$.

Induction Step: To prove that $P(k + 1)$ is true. Consider a set $S$ of $k + 1$ lines such that no two of them are parallel and no three of them pass through a common point. Remove any line $\ell$ from the set $S$. Let $S'$ be the resulting set of $k$ lines. By induction hypothesis, the $k$ lines in $S'$ divide the plane into $(k^2 + k + 2)/2$ regions. Now we add the line $\ell$ to the set $S'$ to obtain the set $S$. Line $\ell$ intersects exactly once with each of the $k$ lines in $S'$. These intersections divide the line $\ell$ into $k + 1$ line segments. Each of these line segments passes through a region and hence $k + 1$ additional regions are created. Hence, the total number of regions formed by $k + 1$ lines is given by

\[
\frac{k^2 + k + 2}{2} + k + 1 = \frac{k^2 + 3k + 4}{2} = \frac{k^2 + 2k + 1 + k + 3}{2} = \frac{(k + 1)^2 + (k + 1) + 2}{2}
\]

Thus $P(k + 1)$ is correct and this completes the proof.