Example. Let $n$ be a non-negative integer. Show that any $2^n \times 2^n$ region with one central square removed can be tiled using L-shaped pieces, where the pieces cover three squares at a time (Figure 1).

Solution. (Attempt 1) Let $R_n$ denote a $2^n \times 2^n$ region. Let $P(n)$ be the property that $R_n$ with one central square removed can be tiled using L-shaped pieces.

Figure 1: A L-tile and an L-tiling of a $2^2 \times 2^2$ region without a square.

Induction Hypothesis: Assume that $P(k)$ is true for some $k > 0$.

Base Case: We want to prove that $P(0)$ is true. This is true because a $1 \times 1$ region with one central square removed requires 0 tiles.

Induction Step: We want to prove that $P(k+1)$ is true, i.e., region $R_{k+1}$ with one central square removed can be tiled using L-shaped pieces. $R_{k+1}$ can be divided into four regions of size $2^k \times 2^k$. Note that the four central corners of $R_{k+1}$ can be covered using one L-shaped tile and one square hole (Figure 2). Each of the four remaining regions has one hole and is of the size $2^k \times 2^k$. By induction hypothesis, these regions can be covered using L-shaped pieces. Thus, since the four disjoint regions can be covered using L-shaped tiles, $R_{k+1}$ without a central square can also be covered using L-shaped tiles.

Our use of induction hypothesis is incorrect as we have assumed that region $R_k$ without a central square (not a corner square) can be covered using L-shaped tiles.

Surprisingly, we can get around this obstacle by proving the following stronger claim.

“For all positive integers $n$, any $R_n$ region with any one square removed can be L-tiled.”

Let $P(n)$ be the property that $R_n$ without one square can be L-tiled.

Induction Hypothesis: Assume that $P(k)$ is true for some $k$. 
Lecture Outline September 25, 2018

Figure 2: Illustration of the two proof attempts.

**Base Case:** We want to prove that \( P(0) \) is true. This is true because a \( 1 \times 1 \) region with one square removed requires 0 tiles.

**Induction Step:** We want to prove that \( P(k+1) \) is true, i.e., region \( R_{k+1} \) without one square that is located anywhere can be L-tiled. Divide \( R_{k+1} \) into four \( R_k \) regions. One of the four \( R_k \) regions that does not have one square can be L-tiled (using induction hypothesis). Each of the other three \( R_k \) regions without the corner square that is located at the center of \( R_{k+1} \) can be L-tiled (using induction hypothesis). By using one more L-tile we can cover the three central squares of \( R_{k+1} \).

---

**Strong Induction.**

For any property \( P \), if \( P(0) \) and \( \forall n \in \mathbb{N}, P(0) \land P(1) \land P(2) \land \cdots \land P(n) \rightarrow P(n+1) \), then \( \forall n \in \mathbb{N}, P(n) \).

**Example.** Prove that if \( n \) is an integer greater than 1 then either \( n \) is a prime or it can be written as a product of primes.

**Solution.** Let \( P(n) \) be “\( n \) can be written as a product of primes”.

**Induction Hypothesis:** Assume that \( P(j) \) is true for \( 1 < j \leq k \).

**Base Case:** We want to show that \( P(2) \) is true. This is clearly true as 2 is a prime.

**Induction Step:** We want to show that \( P(k + 1) \) is true.

**Case I:** \( k + 1 \) is prime. In this case we are done.

**Case II:** \( k + 1 \) is composite. Then,

\[
k + 1 = a \times b, \quad \text{for some } a \text{ and } b \text{ s.t. } 2 \leq a \leq b < k + 1
\]

By induction hypothesis, \( a \) is a prime or it can be written as a product of primes. The same applies to \( b \). Since \( k + 1 = a \times b \), it can be written as a product of primes, namely those primes in the factorization of \( a \) and those in the factorization of \( b \).
Example. Prove that, for any positive integer \( n \), if \( x_1, x_2, \ldots, x_n \) are \( n \) distinct real numbers, then no matter how the parenthesis are inserted into their product, the number of multiplications used to compute the product is \( n - 1 \).

Solution. Let \( P(n) \) be the property that “If \( x_1, x_2, \ldots, x_n \) are \( n \) distinct real numbers, then no matter how the parentheses are inserted into their product, the number of multiplications used to compute the product is \( n - 1 \).”

Induction Hypothesis: Assume that \( P(j) \) is true for all \( j \) such that \( 1 \leq j \leq k \).

Base Case: \( P(1) \) is true, since \( x_1 \) is computed using 0 multiplications.

Induction Step: We want to prove \( P(k + 1) \). Consider the product of \( k + 1 \) distinct factors, \( x_1, x_2, \ldots, x_{k+1} \). When parentheses are inserted in order to compute the product of factors, some multiplication must be the final one. Consider the two terms, of this final multiplication. Each one is a product of at most \( k \) factors. Suppose the first and the second term in the final multiplication contain \( f_k \) and \( s_k \) factors. Clearly, \( 1 \leq f_k, s_k \leq k \). Thus, by induction hypothesis, the number of multiplications to obtain the first term of the final multiplication is \( f_k - 1 \) and the number of multiplications to obtain the second term of the final multiplication is \( s_k - 1 \). It follows that the number of multiplications to compute the product of \( x_1, x_2, \ldots, x_k, x_{k+1} \) is

\[
(f_k - 1) + (s_k - 1) + 1 = f_k + s_k - 1 = k + 1 - 1 = k
\]

Example. The game of NIM is played as follows: Some positive number of sticks are placed on the ground. Two players take turns, removing one, two or three sticks. The player to remove the last stick loses.

A winning strategy is a rule for how many sticks to remove when there are \( n \) left. Prove that the first player has a winning strategy iff the number of sticks, \( n \), is not \( 4k + 1 \) for any \( k \in \mathbb{N} \).

Solution. We will show that if \( n = 4k + 1 \) then player 2 has a strategy that will force a win for him, otherwise, player 1 has a strategy that will force a win for him.

Let \( P(n) \) be the property that if \( n = 4k + 1 \) for some \( k \in \mathbb{N} \) then the first player loses, and if \( n = 4k, 4k + 2, \) or \( 4k + 3 \), the first player wins. This exhausts all possible cases for \( n \).

Induction Hypothesis: Assume that for some \( z \geq 1 \), \( P(j) \) is true for all \( j \) such that \( 1 \leq j \leq z \).

Base Case: \( P(1) \) is true. The first player has no choice but to remove one stick and lose.

Induction Step: We want to prove \( P(z + 1) \). We consider the following four cases.

Case I: \( z + 1 = 4k + 1 \), for some \( k \). We have already handled the base case, so we can assume that \( z + 1 \geq 5 \). Consider what the first player might do to win: he can remove 1, 2, or 3 sticks. If he removes one stick then the remaining number of sticks \( n = 4k \). By strong induction, the player who plays at this point has a winning strategy. So the player who played first loses. Similarly, if the first player removes two sticks or three sticks, the remaining number of sticks is \( 4(k - 1) + 3 \) and \( 4(k - 1) + 2 \) respectively. Again, the first player loses (using induction hypothesis). Thus, in this case, the first player loses regardless of what move he/she makes.

Case II: \( z + 1 = 4k \), or \( z + 1 = 4k + 2 \), or \( z + 1 = 4k + 3 \). If the first player removes three
sticks in the first case, one stick in the second case, and two sticks in the third case then
the second player sees $4(k - 1) + 1$ sticks in the first case and $4k + 1$ sticks in the other two
cases. By induction hypothesis, in each case the second player loses.

**Example.** Prove that the two forms of induction, weak induction and strong induction,
are equivalent. In other words, prove that any statement that admits a strong induction
proof can be proved using weak induction and vice-versa.

**Solution.** Suppose we want to show that a $P(n)$ is true for all positive integers $n \ge n_0$.
The two forms of inductive proofs are as follows.

**Weak Induction:** Assume that

- $(a_w) \ P(n_0)$ is true
- $(b_w)$ For any $k \ge n_0$, $P(k) \implies P(k+1)$ is true.

Then, $P(n)$ is true for all positive integers $n \ge n_0$.

**Strong Induction:** Assume that

- $(a_s) \ P(n_0)$ is true
- $(b_s)$ For any $k \ge n_0$, $P(n_0) \land P(n_0+1) \land \cdots \land P(k) \implies P(k+1)$ is true.

Then, $P(n)$ is true for all positive integers $n \ge n_0$.

We will show that it is always possible to convert a strong induction proof into a weak
induction proof and vice-versa.

The conversion from a weak induction proof to a strong induction proof is trivial, since $(b_s)$
implies $(b_w)$.

We now show that a strong induction proof can be converted to a weak induction proof. Let

$$Q(n) \doteq P(n_0) \land P(n_0+1) \land \cdots \land P(n)$$

**Induction Hypothesis:** Assume that $Q(k)$ is true for some $k \ge n_0$.

**Base Case:** Since $Q(n_0) = P(n_0)$ and we know that $P(n_0)$ is true from $(a_s)$, $Q(n_0)$ is true.

**Induction Step:** We want to show that $Q(k) \implies Q(k+1)$. We have

- $Q(k) \implies P(k+1)$ (from $(b_s)$)
- $\therefore Q(k) \implies Q(k) \land P(k+1)$
- $\therefore Q(k) \implies Q(k+1)$

Thus we have converted a strong induction proof in $P$ to a weak induction proof in $Q$. 