- Office hours
  - 1-2pm ET TODAY

- Meeting with CIS160 alumni
  - 10pm - 11pm ET TODAY (no recording)

Counting.

**Goal:** Find the cardinality of some set.

- Combinatorics
- Ways to arrange objects satisfying some constraints.

In other words, we are interested in finding the cardinality of a set.

Set: unordered collection of objects.

\[ S = \{ 'a', 1, \text{red}, 45 \} \]

Cardinality of a set \( S \) denoted by \( |S| \)
is the # elements in $S$.

$|S| = 4$

Two sets are equal \( \iff \) they have the same cardinality.

\[ A = \{1, 2\}, \quad B = \{a, b\} \]

If two sets are equal then they have the same cardinality.

Two sets are equal \( \iff \) they have exactly the same elements.

\[ A = \{1, 2, a'\}, \quad B = \{a', 2, 1\} \]

\[ A = B \]

Element 2 belongs to

Element 2 $\in$ A.
A set $A$ is a **subset** of set $B$ if every element in $A$ is an element of $B$.

Clearly, $\forall x \in A$, $A \subseteq A$.

Empty set $\emptyset$, $\{\}$, is a subset of every set:

$\forall \text{ set } S$, $\emptyset \subseteq S$.

$A = \{\emptyset\}$, $A' = \{\}$

$A = A'$. $|A| = 1$, $|A'| = 0$

**Subset**: $\forall x \in A$ if $x \in A$ then $x \in B$.

$A = \{1\}$, $B = \{1, 2\}$.
A is a proper subset of B if A ⊂ B and A ≠ B.

Powerset: Powerset \( P(S) \) of a set \( S \)

in the set of all subsets of \( S \).

\( S = \{1, 2, 3\} \)

\( P(S) = \{\{1\}, \{1, 2\}, \{2\}, \{3\}\} \)

\( |S| = 3, \quad |P(S)| = 8. \)

\( |S| = n, \quad |P(S)| = 2^n. \)

Set of integers: \( \mathbb{Z} \)

Positive \( \mathbb{Z}^+ \)
\textbf{real} : \mathbb{R}

\textbf{rational} : \mathbb{Q}

\textbf{natural} : \mathbb{N}

\textbf{set} of all positive even integers \leq 100.
\{2, 4, 6, \ldots, 100\}

\{ x \mid x \in \mathbb{Z}^+ \text{ and } x = 2k, \text{ for some } k \text{ and } 1 \leq x \leq 100 \}

\{ x \mid x = 2k, k \in \mathbb{Z}, 1 \leq k \leq 50 \}

\underline{Theorem}: If \( m \) & \( n \) are integers and \( m \leq n \) then there are \( n-m+1 \) integers from \( m \) to \( n \), inclusive.
Ex: # 3-digit int b/w 100-999 that ar
divisible by 5? 1000

$$\begin{align*}
8th: \quad \frac{900}{5} &= 180 \\
100 & \quad 105 & \quad 110 & \quad \ldots \quad 995 \\
5.20 & \quad 5.21 & \quad 5.22 & \quad 5.199 \\
199 - 20 + 1 &= 180.
\end{align*}$$

Ex: Two teams A, B

- Best of 3 match
- One team wins 2 games.
- # possible outcomes if the match?
If a procedure can be broken up into
k steps s.t.
- Step 1 can be done in \( n_1 \) ways \((2)\)
- Step 2 " " " " \( n_2 \) ways regardless
  of the outcome of the previous step \((2)\)
- Step \( j \) " " " " \( n_j \) ways regardless

\[ \text{Ans: } 6 \]
\[ \text{Ans: } 8 \]
\[ \frac{2}{100} = 0.2 \]
of the outcomes of the previous steps:

\[ \text{Step } k \ldots \ldots \text{ (2)} \]

Then, the number of outcomes of the procedure:

\[ n_1 \times n_2 \times \ldots \times n_k \]  
(Multiplication Rule).

Ex: Chairs of an auditorium
- upper-case letters
- the int \( \leq 100. \)

\[ \# \text{ possible labelings?} \]

Solve: What is the set whose cardinality we are trying to find?
- set of all chair labelings.
\{ \emptyset , A , S , S , C , 19 \ldots \} \\

What is an element of the set?

Chair labeling.

The proc. of constructing a chair label is as follows:

\textbf{S1:} Choose an upper-case letter.

\textbf{S2:} " " thm 1 \leq 100 7.

\#ways to do Step 1 : 26.

" " Step 2 : 101 100

\[ = 26 \times 101 = 2626 \] 9.

Ex: 3 officers: Pres, Treasurer, Sec.
4 people: A, B, C, D.

A cannot be the President.

C or D must be the Sec.

No officer assignments?

Set: $\{B, A, C, \ldots\}$

The proc. to construct an officer assignment is as follows.

$S_1$: Choose the President | 3

$S_2$: "" "" Treas. | 3

$S_3$: "" "" Sec | 2
By the Malt. rule, $\#\text{assign} = 3 \times 3 \times 2 = 18$.

$S_1$: Choose the Sec    | 2
$S_2$: Choose the Trees   | 3
$S_3$: " "  Pres.

$S_1$: Choose the Sec    | 2 ✓
$S_2$: " "  Poes.       | 2 ✓
$S_3$: " "  Trees       | 2 ✓

\[ \prod_{i=1}^{n} \] 

Ex: \( S = \{ x_1, x_2, \ldots, x_n \} \)

\(|P(S)| \) ?
$$S_0 : \rightarrow \{ \{x_1\}, \{x_1, x_2\}, \emptyset, \{x_2, x_3\}\} \cup \{x_4, x_5\}$$

The proc. of constr. a subset \( S \) is as fall.

\( S_1 \): Pick any element.

\( S_2 \): Decide another elem. decide whether \( x_4 \) is in or out

\( S_1 \): Decide the fate of \( x_1 - 2 \)

\( S_2 \): \( \ldots \) \( \ldots \) \( x_2 - 2 \)

\( S_n \): \( \ldots \) \( \ldots \) \( x_n - 2 \)

\( \square \)

\( 2^n \)

Ex: # odd int b/w 1000 & 9999,
that have distinct digits?

\[ \text{Soln: } \{1 \, 2 \, 3 \, 7\} \]

The pec: \( 1 \ \underline{2} \ \underline{3} \ \underline{4} \)

\( \text{b/w 1000 & 9999} \) that have

distinct digits is as foll.

\( \text{S}_1: \) Choose the last digit: \( 5 \)

\( \text{S}_2: \) " " first: \( 8 \)

\( \text{S}_3: \) 2nd: \( 8 \)

\( \text{S}_4: \) 3rd: \( 7 \)

\( \text{Ans: } 5 \times 8 \times 8 \times 7. \)
Permutation, order.

\[ S = \{ a, b, c \} \]

All possible permutation/orders of elements in S.

\[ \begin{align*}
    a & \quad b & \quad c \\
    a & \quad c & \quad b \\
    b & \quad a & \quad c \\
    b & \quad c & \quad a \\
    c & \quad a & \quad b \\
    c & \quad b & \quad a
\end{align*} \]

\[ \begin{array}{c}
    6.
\end{array} \]

\[ S = \{ x_1, x_2, \ldots, x_n \} \]
# permutation of elem in S?
The proc. to constr. a
permutation of elem in S is
as follows:

$\begin{array}{cccc}
1 & 2 & 3 & \cdots & n \\
\end{array}$

$S_1$: Choose an elem for pos. 1
$S_2$: " " " " " 2

# ways to do step 1: n

2 : n-1
\[ n = 1 \]

By the MR: \( 1 \times 2 \times \ldots \times n \)

\[ = n! \]

**Ex:** \( \{a,b,c,d,e,f,g,h\} \)

# permutations that contain the string "abc".

\( \{ abc, d, e, f, g, h \} \)

\[ \frac{6!}{6!} \]

the move I constr. \ldots \ldots
$S_1$: Choose a pose for a \( f \)  \\
$S_2$:  \\

6! \\

--- \\
\( a \times \tau_a \)