Introduction to Logic

A *proposition* is a statement that is either true or false. For example, “2 + 2 = 4” and “Donald Knuth is a faculty at Rutgers-Camden” are propositions, whereas “What time is it?”, $x^2 < x + 40$ are not propositions.

We can construct compound propositions from simpler propositions by using some of the following connectives. Let $p$ and $q$ be arbitrary propositions.

**Negation:** $\neg p$ (read as “not $p$”) is the proposition that is true when $p$ is false and vice-versa.

**Conjunction:** $p \land q$ (read as “$p$ and $q$”) is the proposition that is true when both $p$ and $q$ are true.

**Disjunction:** $p \lor q$ (read as “$p$ or $q$”) is the proposition that is true when at least one of $p$ or $q$ is true.

**Exclusive Or:** $p \oplus q$ (read as “$p$ exclusive-or $q$”) is the proposition that is true when exactly one of $p$ and $q$ is true is false otherwise.

**Implication:** $p \to q$ (read as “$p$ implies $q$”) is the proposition that is false when $p$ is true and $q$ is false and is true otherwise.

The implication $q \to p$ is called the *converse* of the implication $p \to q$. The implication $\neg p \to \neg q$ is called the *inverse* of $p \to q$. The implication $\neg q \to \neg p$ is the *contrapositive* of $p \to q$. *$p$ only if $q$* means “if not $q$ then not $p$”, or equivalently if $p$ then $q$.

**Biconditional:** $p \iff q$ (read as “$p$ if, and only if, $q$”) is the proposition that is true if $p$ and $q$ have the same truth values and is false otherwise. “If and only if” is often abbreviated as iff.

The following truth table makes the above definitions precise.

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<thead>
<tr>
<th>$p$</th>
<th>$q$</th>
<th>$\neg p$</th>
<th>$p \land q$</th>
<th>$p \lor q$</th>
<th>$p \oplus q$</th>
<th>$p \to q$</th>
<th>$q \to p$</th>
<th>$p \iff q$</th>
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**Necessary and Sufficient Conditions:** For propositions $p$ and $q$,
p is a **sufficient** condition for q means that $p \rightarrow q$.

p is a **necessary** condition for q means that $\neg p \rightarrow \neg q$, or equivalently $q \rightarrow p$.

Why is $p \land q$ not the correct answer?

Thus $p$ is a necessary and sufficient condition for q means “p iff q”.

**Logical Equivalence**

Two compound propositions are logically equivalent if they always have the same truth value. Two statement $p$ and $q$ can be proved to be logically equivalent either with the aid of truth tables or using a sequence of previously derived logically equivalent statements.

**Example.** Show that $p \rightarrow q \equiv \neg p \lor q \equiv \neg q \rightarrow \neg p$.

**Solution.** The truth table below proves the above equivalence.

<table>
<thead>
<tr>
<th>$p$</th>
<th>$q$</th>
<th>$\neg p$</th>
<th>$\neg q$</th>
<th>$p \rightarrow q$</th>
<th>$\neg p \lor q$</th>
<th>$\neg q \rightarrow \neg p$</th>
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**Example.** Show that $p \equiv \neg p \rightarrow C$ and $p \rightarrow q \equiv (p \land \neg q) \rightarrow C$.

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<tr>
<th>$p$</th>
<th>$q$</th>
<th>$\neg p$</th>
<th>$\neg q$</th>
<th>$p \rightarrow q$</th>
<th>$p \land \neg q$</th>
<th>$\neg p \rightarrow C$</th>
<th>$(p \land \neg q) \rightarrow C$</th>
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The above equivalence forms the basis of proofs by contradiction.

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**The logic of Quantified Statements**

Consider the statement $x < 15$. We can denote such a statement by $P(x)$, where $P$ denotes the predicate “is less than 15” and $x$ is the variable. This statement $P(x)$ becomes a proposition when $x$ is assigned a value. In the above example, $P(8)$ is true while $P(18)$ is false.

Another way to convert the statement $P(x)$ into a proposition is through **quantification**. The two types of quantification that we will study are **universal quantification** and **existential quantification**. Using universal quantifier $\forall$ (“for all”) alongside $P(x)$ means that the statement $P(x)$ is true for all elements in the domain of $x$. Thus the proposition $\forall x \in D, P(x)$ is true when $P(x)$ is true for all $x \in D$ and is false if there is an element
$x' \in D$ for which $P(x')$ is false. Using existential quantifier $\exists$ (“there exists”) alongside $P(x)$ means that there exists an element in the domain of $x$ for which $P(x)$ is true. Thus the proposition $\exists x \in D, P(x)$ is true if there is an $x' \in D$ for which $P(x')$ is true and is false if $P(x)$ is false for all $x \in D$.

Examples of propositions using quantifiers are as follows.

1. $\forall x \in \mathbb{Z}, x^3 + 1$ is composite.
2. $\forall x \in \mathbb{Z}, x$ is even $\rightarrow x + 1$ is odd.
3. $\exists x \in \mathbb{N}, x^2 \neq x$.
4. $\exists x \in \mathbb{Z}, 2|x$ and $2|x + 1$.
5. $\forall x \in \mathbb{Z}, \exists y \in \mathbb{Z}, x + y = 0$.
6. $\exists x \in \mathbb{Z}, \forall y \in \mathbb{Z}, x > y$.

Sometimes it helps (in proofs) to consider the negation of a proposition. Verify the following equivalence.

$$\neg(\forall x \in D, P(x)) \equiv \exists x \in D, \neg P(x)$$

$$\neg(\exists x \in D, P(x)) \equiv \forall x \in D, \neg P(x)$$

**Proofs**

We will illustrate some proof techniques by proving some properties about numbers. Before we do that let’s go through some basic definitions given below.

An integer $n$ is **even** iff $n = 2k$ for some integer $k$. An integer is **odd** iff $n = 2k + 1$ for some integer $k$. Symbolically,

- $n$ is even $\iff \exists$ an integer $k$ s.t. $n = 2k$
- $n$ is odd $\iff \exists$ an integer $k$ s.t. $n = 2k + 1$

An integer $n$ is **prime** iff $n > 1$ and for all positive integers $r$ and $s$, if $n = r \cdot s$, then $r = 1$ or $s = 1$. Otherwise $n$ is **composite**.

Given any real number $x$, the **floor of** $x$, denoted by $\lfloor x \rfloor$, is defined as follows

$$\lfloor x \rfloor = n \iff n \leq x < n + 1,$$  where $n$ is an integer

Given any real number $x$, the **ceiling of** $x$, denoted by $\lceil x \rceil$, is defined as follows

$$\lceil x \rceil = n \iff n - 1 < x \leq n,$$  where $n$ is an integer

A real number is **rational** iff it can be expressed as a ratio of two integers with a non-zero denominator. A real number that is not rational is **irrational**. More formally,

$$r \text{ is rational } \iff \exists \text{ integers } a \text{ and } b \text{ such that } r = a/b \text{ and } b \neq 0.$$
Example. Prove the following: If the sum of two integers is even then so is their difference.

Solution. Let $m$ and $n$ be particular but arbitrarily chosen integers such that $m + n$ is even. By definition of even, we have $m + n = 2k$, for some integer $k$. Then

$$m = 2k - n$$

Now $m - n$ can be written as follows.

$$m - n = 2k - n - n$$

$$= 2(k - n)$$

Since $k$ and $n$ are integers, $k - n$ is an integer, $2(k - n)$ is even and hence $m - n$ is even.

Example. Prove that, for all integers $n$, if $n$ is odd then $n^2 + n + 1$ is odd.

Solution. Since $n$ is odd $n = 2k + 1$ for some integer $k$. Then,

$$n^2 + n + 1 = (2k + 1)^2 + 2k + 1 + 1$$

$$= 4k^2 + 4k + 1 + 2k + 2$$

$$= 4k^2 + 6k + 2 + 1$$

$$= 2(2k^2 + 3k + 1) + 1$$

Since $k$ is an integer, $p = 2k^2 + 3k + 1$ is an integer and $n^2 + n + 1$ is odd, since $n^2 + n + 1 = 2p + 1$ where $p$ is an integer.