OH TODAY
- 10 pm - 11 pm.

Ex: Let $n$ be an integer. If $n > 1$ then

$n^3 + 1$ is composite.

Proof: Let $n$ be an arbitrary, but

particular integer s.t. $n > 1$. Then

$n^3 + 1 = n^3 (1 + \frac{1}{n^3})$

Since $n > 1$, $n^3$ is clearly greater than 1.

Since $\frac{1}{n^3} > 0$, $1 + \frac{1}{n^3} > 1$ &

hence $(n^3 + 1)$ is composite. X Begin!

$n^3 + 1 = (n+1)(n^2-n+1)$
Clearly, both terms are addition, subtraction, & prod. I & t & hence are integers.

\(x + 1\) is clearly greater than 1, since \(x > 1\).

What remains to show is that

\[x^2 - x + 1 > 0\]

\[x > 1\]

\[x^2 > x\]

\[x^2 - x > 0\]

\[x^2 - x + 1 > 1\]

\textbf{Done!}

\textbf{Ex:} Prove that if \(x\) and \(y\) are integers where \(x + y\) is even, then \(x\) and \(y\) are both odd or both even. (Want to prove \(p \Rightarrow q\) and \(q \Rightarrow p\).)
Prop: We will prove the claim by proving its contrapositive. That is, we will show that if exactly one of \( x \) or \( y \) is even and the other is odd, then \( x + y \) is odd. Without loss of generality, let:

Case I: \( x \) is even and \( y \) is odd.

By def., let \( x = 2k \), for some \( k \).

And \( y = 2l + 1 \), for some \( l \).

\[
x + y = 2k + 2l + 1
\]

\[
= 2(k + l) + 1
\]

\[
= 2z + 1, \quad \text{where } z = k + l
\]

\[
\therefore x + y \text{ is odd, by def.}
\]

Case II: \( x \) is odd and \( y \) is even.
Ex: Show that at least three of any 25 days chosen must fall in the same month of the year.

**Proof**: We want to show that if 25 days are chosen then \( \geq 3 \) must fall in the same month of the year.

\[
\begin{align*}
\neg (p \Rightarrow q) & \equiv \neg (\neg p \Rightarrow \neg q) \\
& \equiv \neg \neg p \land q \\
& \equiv p \land q
\end{align*}
\]

Negation: \( \neg (p \lor q) = p \land \neg q \).

Assume for contradiction that we have chosen 25 days and \( \leq 2 \) days fall in any month. We know that
Thus are 12 months in a year & hence a total of \( \leq 12 \times 2 = 24 \) days an year. This is a contradiction!

Ex: Prove that if \( 3n + 2 \) is odd then \( n \) is odd.

Proof: Assume for contradiction that

\( 3n + 2 \) is odd and \( n \) is even.

By defn, \( n = 2k \), for some int \( k \).

\[
\begin{align*}
3n + 2 &= 3(2k) + 2 \\
&= 2(3k + 1) \\
&= 2 \cdot l, \text{ where } l = 3k + 1 \text{ is an int.}
\end{align*}
\]

\( \therefore (3n + 2 \text{ is an even no.}) \)
Thus \((3n+2 \text{ is odd}) \land (3n+2 \text{ is even})\) is clearly a contradiction, hence our original claim must be true.

Ex: Prove that for all real nos \(a \& b\), if \(ab\) is irrational then either \(a\) or \(b\) or both must be irrational.

Proof: We will prove the claim by proving its contrapositive. That is, we will show that if \(ab\) is rational then exactly both \(a\) \& \(b\) must be rational.
That is we will show that if both \( a \) & \( b \) are rational then \( ab \) is rational. By defn of rational nos., let
\[
a = \frac{p}{q} \quad \text{&} \quad b = \frac{r}{s},
\]
where
\( p, q, r, s \) are int & \( q \neq 0 \) & \( s \neq 0 \).

\[
\therefore ab = \frac{pr}{qs}.
\]

Clearly, \( pr \) & \( qs \) are int & since
\( q \neq 0 \) & \( s \neq 0 \), \( qs \neq 0 \). Thus

\( ab \) is a rational no.

A & B are sets.
\[ A \cup B = \{ x \mid x \in A \cup x \in B \} \]

\[ A \cap B = \{ x \mid x \in A \cap x \in B \} \]

Two sets are disjoint iff \( A \cap B = \emptyset \).

\[ A = \{1, 2\}, \quad B = \{1, 3\} \]

\[ A \setminus B = \{1, 2\} \]

\[ A \cup B = \{1, 2, 3\} \]

\[ A \cap B = \{1\} \]

A collection of non-empty sets \( \{A_1, A_2, \ldots, A_n\} \)

partition the set \( A \) iff

(i) \( A_1 \cup A_2 \cup \ldots \cup A_n = A \).

(ii) \( A_1, A_2, \ldots, A_n \) are pairwise disjoint.
Set difference: $A \setminus B$

Cartesian product: $A \times B = \{ (a, b) \mid a \in A \text{ and } b \in B \}$

$A = \{1, 2\}$

$B = \{1, c\}$

$A \times B = \{(1,1), (1,c), (2,1), (2,c)\}$

$A = \{1, 2\}$, $B = \{1, c\}$
\[ A \times B = \{ (1, 3), (2, 3) \}. \]

\[ |A| = 2, \quad |B| = 3. \]

\[ |A \times B| = 5, \quad 4, \quad 3, \quad 10. \]

\[ 6 \checkmark. \]

**Ex:** Let \( A = \{ 2, 2^2, 2^3, \ldots \} \)

\[ B = \{ 2, 4, 6, \ldots \}. \] Prove that \( A \subseteq B \)

**Proof:** Note that

\[ B = \{ 2i \mid i \in \mathbb{Z}^+ \}. \]

Let \( x \) be an arbitrary but particular element in \( A \). Let

\[ x = 2^k, \] for some int \( k > 1. \)
\[ = 2 \cdot \left[ \begin{array}{c} k-1 \\ 2 \end{array} \right] \]

Since \( k \geq 1 \), \( k-1 \geq 0 \) & hence \( 2^{k-1} \geq 1 \). Thus, \( x = 2 \cdot j \),
where \( j \in \mathbb{Z}^+ \) & hence \( x \in B \).

**Ex:** Let \( A \) & \( B \) be sets. Then
\[ A = B \iff A \subseteq B \text{ and } B \subseteq A. \]

_Proof:_
\[ (\Rightarrow) \text{ A } = \text{ B } \Rightarrow A \subseteq B \text{ and } B \subseteq A. \]
\[ A \subseteq A \text{ & since A } = \text{ B }, A \subseteq B \text{ and } B \subseteq A. \]
\[ \text{ B } \subseteq A. \]

\[ (\Leftarrow) \boxed{A \subseteq B} \text{ and } \boxed{B \subseteq A} \Rightarrow A = B. \]

Let \( x \) be any element in \( A \).

Since \( A \subseteq B \), clearly \( x \in B \).
Since \( B \subseteq A \), there is no element in \( B \) that is not in \( A \).

Similarly, if \( y \) is any element in \( B \), we can show that \( y \in A \) & there is no element in \( A \) that is not in \( B \). Thus both \( A \) & \( B \) must have the same elements.

Ex: Prove that if \( A \) & \( B \) are non-empty set then

\[
A \times B = B \times A \quad \text{if} \quad A = B.
\]

**Proof:** \((\Leftrightarrow) \) \( A = B \Rightarrow A \times B = B \times A \).
AxB = AxA = BxA.

(⇒) AxB = BxA ⇒ A = B.

let x & y be arb. but particular elements in A & B resp. We want to show that x ∈ B and y ∈ A.

Clearly, (x, y) ∈ AxB. Since

AxB = BxA , (x, y) ∈ BxA.

This mean that x ∈ B & y ∈ A.