Example. Prove that the product of two odd numbers is an odd number.

Solution. Let $x$ and $y$ be particular but arbitrarily chosen odd numbers. Then, $x = 2k + 1$ and $y = 2l + 1$, for some integers $k$ and $l$. We have

$$x \cdot y = (2k + 1) \cdot (2l + 1) = 4kl + 2(k + l) + 1 = 2(2kl + k + l) + 1$$

Let $p = 2kl + k + l$. Since $k$ and $l$ are integers, $p$ is an integer and $x \cdot y = 2p + 1$ is odd.

Example. Prove that $\sqrt{2}$ is irrational.

Solution. For the purpose of contradiction, assume that $\sqrt{2}$ is a rational number. Then there are integers $a$ and $b$ ($b \neq 0$) with no common factors such that

$$\sqrt{2} = \frac{a}{b}$$

Squaring both sides of the above equation gives

$$2 = \frac{a^2}{b^2} \quad \Rightarrow \quad a^2 = 2b^2 \quad (1)$$

From (1) we conclude that $a^2$ is even. This fact combined with the result of previous example implies that $a$ is even. Then, for some integer $k$, let $a = 2k \quad (2)$

Combining (1) and (2) we get

$$4k^2 = 2b^2$$

$$2k^2 = b^2$$

The above equation implies that $b^2$ is even and hence $b$ is even. Since we know $a$ is even this means that $a$ and $b$ have 2 as a common factor which contradicts the assumption that $a$ and $b$ have no common factors.

We will now give a very elegant proof for the fact that “$\sqrt{2}$ is irrational” using the unique factorization theorem which is also called the fundamental theorem of arithmetic.

The unique factorization theorem states that every positive number can be uniquely represented as a product of primes. More formally, it can be stated as follows.
Given any integer \( n > 1 \), there exist a positive integer \( k \), distinct prime numbers \( p_1, p_2, \ldots, p_k \), and positive integers \( e_1, e_2, \ldots, e_k \) such that
\[
n = p_1^{e_1} p_2^{e_2} p_3^{e_3} \cdots p_k^{e_k}
\]
and any other expression of \( n \) as a product of primes is identical to this except, perhaps, for the order in which the factors are written.

Example. Prove that \( \sqrt{2} \) is irrational using the unique factorization theorem.

Solution. Assume for the purpose of contradiction that \( \sqrt{2} \) is rational. Then there are integers \( a \) and \( b \) (\( b \neq 0 \)) such that
\[
\sqrt{2} = \frac{a}{b}
\]
Squaring both sides of the above equation gives
\[
2 = \frac{a^2}{b^2}
\]
Let \( S(m) \) be the sum of the number of times each prime factor occurs in the unique factorization of \( m \). Note that \( S(a^2) \) and \( S(b^2) \) is even. Why? Because the number of times that each prime factor appears in the prime factorization of \( a^2 \) and \( b^2 \) is exactly twice the number of times that it appears in the prime factorization of \( a \) and \( b \). Then, \( S(2b^2) = 1 + S(b^2) \) must be odd. This is a contradiction as \( S(a^2) \) is even and the prime factorization of a positive integer is unique.

Example. Prove or disprove that the sum of two irrational numbers is irrational.

Solution. The above statement is false. Consider the two irrational numbers, \( \sqrt{2} \) and \( -\sqrt{2} \). Their sum is \( 0 = 0/1 \), a rational number.

Example. Show that there exist irrational numbers \( x \) and \( y \) such that \( x^y \) is rational.

Solution. We know that \( \sqrt{2} \) is an irrational number. Consider \( \sqrt{2}^{\sqrt{2}} \).

Case I: \( \sqrt{2}^{\sqrt{2}} \) is rational.
In this case we are done by setting \( x = y = \sqrt{2} \).

Case II: \( \sqrt{2}^{\sqrt{2}} \) is irrational.
In this case, let \( x = \sqrt{2}^{\sqrt{2}} \) and let \( y = \sqrt{2} \). Then, \( x^y = \left(\sqrt{2}^{\sqrt{2}}\right)^{\sqrt{2}} = (\sqrt{2})^2 = 2 \), which is an integer and hence rational.
Example. Prove that for all positive integers $n$,

$$n \text{ is even } \leftrightarrow 7n + 4 \text{ is even}$$

Solution. Let $n$ be a particular but arbitrarily chosen integer.

*Proof for $n$ is even $\rightarrow 7n + 4$ is even.* Since $n$ is even, $n = 2k$ for some integer $k$. Then,

$$7n + 4 = 7(2k) + 4 = 2(7k + 2)$$

Hence, $7n + 4$ is even.

*Proof for $7n + 4$ is even $\rightarrow n$ is even.* Since $7n + 4$ is even and $n$ is a positive integer, let $7n + 4 = 2l$ for some integer $l \geq 6$. Then,

$$7n = 2l - 4 = 2(l - 2)$$

Clearly, $7n$ is even. Combining the fact that 7 is odd with the result of the Example 1, we conclude that $n$ is even.

We can also prove the latter by proving its contrapositive, i.e., we can prove

if $n$ is odd then $7n + 4$ is odd.

Since $n$ is a positive odd integer, we have $n = 2k + 1$, for some integer $k \geq 0$. Thus we have

$$7n + 4 = 7(2k + 1) + 4 = 14k + 10 + 1 = 2(7k + 5) + 1 = 2k' + 1, \text{ where } k' = 7k + 5 \text{ is an integer.}$$

---

Example. Prove that there are infinitely many prime numbers.

Solution. Assume, for the sake of contradiction, that there are only finitely many primes. Let $p$ be the largest prime number. Then all the prime numbers can be listed as

$$2, 3, 5, 7, 11, 13, \ldots, p$$

Consider an integer $n$ that is formed by multiplying all the prime numbers and then adding 1. That is,

$$n = (2 \times 3 \times 5 \times 7 \times \cdots p) + 1$$

Clearly, $n > p$. Since $p$ is the largest prime number, $n$ cannot be a prime number. In other words, $n$ is composite. Let $q$ be any prime number. Because of the way $n$ is constructed, when $n$ is divided by $q$ the remainder is 1. That is, $n$ is not a multiple of $q$. This contradicts
the Fundamental Theorem of Arithmetic.

Alternate Proof by Filip Saidak. Let \( n \) be an arbitrary positive integer greater than 1. Since \( n \) and \( n + 1 \) are consecutive integers, they must be relatively prime. Hence, the number \( N_2 = n(n + 1) \) must have at least two different prime factors. Similarly, since the integers \( n(n + 1) \) and \( n(n + 1) + 1 \) are consecutive, and therefore relatively prime, the number

\[
N_3 = n(n + 1)[n(n + 1) + 1]
\]

must have at least three different prime factors. This process can be continued indefinitely, so the number of primes must be infinite.

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Mathematical Induction

Example. Prove that for all integers \( n \geq 1 \),

\[
\sum_{i=1}^{n} i = \frac{n(n + 1)}{2}
\]

Solution. We will prove the claim using induction on \( n \).

Induction hypothesis: Assume that the claim is true when \( n = k \), for some \( k \geq 1 \). In other words assume that

\[
\sum_{i=1}^{k} i = \frac{k(k + 1)}{2}
\]

Base Case: \( n = 1 \). The claim is true for \( n = 1 \) as both sides of the equation equal to 1.

Induction step: To prove that the claim is true when \( n = k + 1 \). That is, we want to show that

\[
\sum_{i=1}^{k+1} i = \frac{(k + 1)(k + 2)}{2}
\]

We can do this as follows.

\[
\sum_{i=1}^{k+1} i = \sum_{i=1}^{k} i + (k + 1) = \frac{k(k + 1)}{2} + k + 1 \quad \text{(using induction hypothesis)}
\]

\[
= \frac{k(k + 1) + 2(k + 1)}{2}
\]

\[
= \frac{(k + 1)(k + 2)}{2}
\]
Example. Prove that the sum of the first $n$ positive odd numbers is $n^2$.

Solution. We want to prove that $\forall$ positive integers $n, P(n)$ where $P(n)$ is the following property.

$$\sum_{i=0}^{n-1} 2i + 1 = n^2$$

Base Case: We want to show that $P(1)$ is true. This is clearly true as

$$\sum_{i=0}^{0} 2i + 1 = 1 = 1^2$$

Induction Hypothesis: Assume $P(k)$ is true for some $k \geq 1$.

Induction Step: We want to show that $P(k+1)$ is true, i.e., we want to show that

$$\sum_{i=0}^{k} 2i + 1 = (k+1)^2$$

We can do this as follows.

$$\sum_{i=0}^{k} 2i + 1 = \sum_{i=0}^{k-1} 2i + 1 + 2k + 1$$

$$= k^2 + 2k + 1 \quad \text{(using induction hypothesis)}$$

$$= (k+1)^2$$