OFFICE HOURS TODAY
- 1-2pm ET.

EE: Prove that every sequence of $n^2+1$ distinct real numbers $x_1, x_2, \ldots, x_{n^2+1}$ contains a subsequence of length $n+1$ that is either strictly increasing or strictly decreasing.

**Example:** $n = 3$

$m_1 = 3$, $m_2 = 3$, $m_3 = 2$, $m_4 = 3$, $m_5 = 2$, $m_6 = 2$, $m_7 = 1$, $m_8 = 2$

\[x_1, x_2, x_3, x_4, 18, 17, 45, 14, 37, 6, 51, 4, 28, 20\]

**Proof:** Suppose there is no subsequence of length $n+1$. We want to then show that there is a subsequence of length $n+1$. Let $m_k$ be the length of the longest subseq. starting from $x_k$.

Thus we have $m_1, m_2, \ldots, m_{n^2+1}$. Note that
1 \leq m_j \leq n, \forall j. We apply PHP as follows.

Consider n holes: one corr. to each value of m_j. The pigeon are m_1, m_2, \ldots, m_{n+1}.

By the PHP, there must be a bin that has \geq n+1 not. That is, \exists m_j's have the same value. Let there nos be

\[ m_{k_1} = m_{k_2} = m_{k_3} = \ldots = m_{k_{n+1}}. \]

\[ 1 \leq k_1 < k_2 < k_3 < \ldots < k_{n+1} \]

Consider the corresponding x_i's

\[ x_{k_1} \geq x_{k_2} > x_{k_3} > \ldots > x_{k_{n+1}} \]

Suppose <
There is a subseq. starting from \( x_k \) that has length \( m_{k_2} + 1 = m_{k_1} + 1 \), a contradiction.

**Probability**

- Random process / experiment
- Mathematical model to represent the experiment
- Sample space \( \Omega \): set of all possible outcomes of the experiment
- Probability distribution: assigning probabilities to each outcome of the experiment, i.e., assigning prob to each outcome \( \Omega \).
- For any event $E$, $\Pr[E] = \sum_{\omega \in E} \Pr[\omega]$

**Example**: Biased coin
- $\Pr[H] = \frac{1}{3}$

Flip the coin twice

$\Pr[\text{we obtain one T and one H}]$?

**Solution**:
$E = \{\text{HT}, \text{TH}\}$

$\Pr[E] = \Pr[\text{HT}] + \Pr[\text{TH}] = \frac{2}{9} + \frac{2}{9} = \frac{4}{9}$.

Ex: Roll two dice. Compute the probability that the two faces are equal if the two dice are distinct.

Solu: $\Omega = \{(1,1), (1,2), \ldots, (1,6), (2,1), (2,2), \ldots, (2,6), \ldots, (6,1), (6,2), \ldots, (6,6)\}$

$\forall \omega \in \Omega, \ Pr[\omega] = \frac{1}{36}$

$E = \{(1,1), (2,2), \ldots, (6,6)\}$
\( \Pr \{ E \} = \frac{1}{36} \cdot 6 = \frac{1}{6} \).

What if the two dice are identical?

\( \Omega = \{ \{1,1\}, \{1,2\}, \{1,3\}, \ldots, \{1,6\}, \{2,2\}, \ldots, \{2,6\}, \ldots, \{3,6\} \} \)

\( X \) Bofus!

Non-uniform distribution.

\(|\Omega| = 21\)

\( \forall \omega \in \Omega, \Pr \{ \omega \} = \frac{1}{21} \).

\( \Pr \{ E \} = \frac{6}{21} = \frac{2}{7} \).

\( \Pr \{ E \} = \frac{6}{36} = \frac{1}{6} \).

Ex: \( \frac{m}{n} \) distinct balls in \( n \) bins. Each ball is equally
no bound on the # balls in a bin.

Pr [ bin 1 contains all the balls ] ?

\[ \Omega = \left\{ (w_1, w_2, \ldots, w_m) \mid w_i : \text{bin that ball } i \text{ lands in} \right\}, \quad 1 \leq w_i \leq n. \]

\[ |\Omega| = n^m \]

\[ \forall \omega \in \Omega, \quad \text{Pr}[\omega] = \frac{1}{n^m}. \]

\[ E = \{ (1, 1, \ldots, 1) \} \]

\[ \text{Pr}(E) = \frac{1}{n^m} \]

Ex: Roll a 6-sided 6 times. What is the prob. of seeing all nos?

\[ \Omega = \left\{ (w_1, w_2, \ldots, w_6) \mid w_i : \text{result of the } i\text{th die} \right\} \]

\[ \text{\color{red}{likely to run out of time.}} \]
\[ |E| = 6 \]

\[ \forall \omega \in \Omega, \quad \Pr(\omega) = \frac{1}{6} \]

\[ E = \{ 1, 2, 3, 4, 5, 6 \} \]

\[ \Pr(E) = \frac{1}{6} \]

\[ \Pr(\bar{E}) = 6! \cdot \frac{1}{6} \]

**Example:**

1. Contestant
2. Can.
3. Contestant

Do you want to switch?

1000 doors
Prize behind one door
999 doors w/goats
Host opens 998 other doors
Should you switch?
Q: Should the contestant switch or stay put?

\[ S = \{ (P, C, h) \} \]

\[ \begin{align*}
\text{pr in door} & \quad \text{door that contestant chose} & \quad \text{door that the host opened} \\
(1, 1, 2) & \quad \frac{1}{3} & \quad \frac{1}{2} & \quad \frac{1}{6} \\
(1, 1, 3) & \quad \frac{1}{3} & \quad \frac{1}{3} & \quad \frac{1}{9} \\
(1, 2, 3) & \quad \frac{1}{3} & \quad \frac{1}{3} & \quad \frac{1}{9} \\
& \frac{1}{9} & \frac{1}{9} & \frac{1}{9} \\
& \frac{1}{15} & \frac{1}{15} & \frac{1}{15} \\
& \frac{1}{18} & \frac{1}{18} & \frac{1}{18} \\
& \frac{1}{9} & \frac{1}{9} & \frac{1}{9} \\
& \frac{1}{18} & \frac{1}{18} & \frac{1}{18} \\
& \frac{1}{18} & \frac{1}{18} & \frac{1}{18} \\
& \frac{1}{18} & \frac{1}{18} & \frac{1}{18} \\
\end{align*} \]

\[ 6 \cdot \frac{1}{9} = \frac{2}{3} \]

always switch

\[ \Pr[ \text{contestant switches & wins} ] = \Pr[ \text{contestant chooses a goat door the 1st time} ] \]

\[ = \frac{2}{3} \]

Inclusion-Exclusion.
\[
\Pr \left( X \cup Y \right) = \sum_{\omega \in X \cup Y} \Pr \left( \omega \right)
\]
\[
\leq \sum_{\omega \in X} \Pr \left( \omega \right) + \sum_{\omega \in Y} \Pr \left( \omega \right) - \sum_{\omega \in X \cap Y} \Pr \left( \omega \right)
\]
\[
\leq \Pr \left( X \right) + \Pr \left( Y \right) - \Pr \left( X \cap Y \right).
\]
\[
\Pr \left( X \cup Y \cup Z \right) \leq \Pr \left( X \cup \Pr \left( Y \right) \cup \Pr \left( Z \right) \right)
\]
\[
- \Pr \left( X \cap Y \right) - \Pr \left( Y \cap Z \right) - \Pr \left( X \cap Z \right)
\]
\[
+ \Pr \left( X \cap Y \cap Z \right).
\]
\[
\Pr \left( \bigcup_{i=1}^{n} X_i \right) \leq \sum_{i=1}^{n} \Pr \left( X_i \right)
\]
Ex: 3 flips of a fair coin.

\[ \Pr \left[ \text{1st flip gives H or the 3rd flip gives a H} \right] \]

Soh: \( \Omega = \{ HHH, HHT, \ldots , THT, \ldots \} \)

\( F_1 \): event that 1st flip gives H.

\( F_2 \): 3rd flip gives H.

\[ \Pr \left[ F_1 \cup F_2 \right] = \Pr \left( F_1 \right) + \Pr \left( F_2 \right) - \Pr \left( F_1 \cap F_2 \right) \]

\[ = \frac{1}{2} + \frac{1}{2} - \frac{1}{4} \]

\[ = \frac{3}{4} \]

Ex: Coin tossed 10 times. \( \Pr \left[ 8 \text{ or more heads occur} \right] \).

Soh: Ex: event that exactly 8 heads
\[ P(\{E_8 \lor E_9\}) = 0 \]

Ex: 8 dice. \[ P(\text{on \geq 1 the \text{ die result is a 4}}) \]