Example. Let $n$ be a non-negative integer. Show that any $2^n \times 2^n$ region with one central square removed can be tiled using L-shaped pieces, where the pieces cover three squares at a time (Figure 1).

Solution. (Attempt 1) Let $R_n$ denote a $2^n \times 2^n$ region. Let $P(n)$ be the property that $R_n$ with one central square removed can be tiled using L-shaped pieces.

Induction Hypothesis: Assume that $P(k)$ is true for some $k \geq 0$.
Base Case: We want to prove that $P(0)$ is true. This is true because a $1 \times 1$ region with one central square removed requires 0 tiles.
Induction Step: We want to prove that $P(k + 1)$ is true, i.e., region $R_{k+1}$ with one central square removed can be tiled using L-shaped pieces.
$R_{k+1}$ can be divided into four regions of size $2^k \times 2^k$. Note that the four central corners of $R_{k+1}$ can be covered using one L-shaped tile and one square hole (Figure 2). Each of the four remaining regions has one hole and is of the size $2^k \times 2^k$. By induction hypothesis, these regions can be covered using L-shaped pieces. Thus, since the four disjoint regions can be covered using L-shaped tiles, $R_{k+1}$ without a central square can also be covered using L-shaped tiles.
Our use of induction hypothesis is incorrect as we have assumed that region $R_k$ without a central square (not a corner square) can be covered using L-shaped tiles.

Surprisingly, we can get around this obstacle by proving the following stronger claim.
“For all positive integers $n$, any $R_n$ region with any one square removed can be L-tiled.”

Let $P(n)$ be the property that $R_n$ without one square can be L-tiled.
Induction Hypothesis: Assume that $P(k)$ is true for some $k$. 
Figure 2: Illustration of the two proof attempts.

**Base Case:** We want to prove that $P(0)$ is true. This is true because a $1 \times 1$ region with one square removed requires 0 tiles.

**Induction Step:** We want to prove that $P(k+1)$ is true, i.e., region $R_{k+1}$ without one square that is located anywhere can be L-tiled. Divide $R_{k+1}$ into four $R_k$ regions. One of the four $R_k$ regions that does not have one square can be L-tiled (using induction hypothesis). Each of the other three $R_k$ regions without the corner square that is located at the center of $R_{k+1}$ can be L-tiled (using induction hypothesis). By using one more L-tile we can cover the three central squares of $R_{k+1}$.

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**Strong Induction.**

For any property $P$, if $P(0)$ and $\forall n \in \mathbb{N}, P(0) \land P(1) \land P(2) \land \cdots \land P(k) \rightarrow P(k+1)$, then $\forall n \in \mathbb{N}, P(n)$.

**Example.** Prove that if $n$ is an integer greater than 1 then either $n$ is a prime or it can be written as a product of primes.

**Solution.** Let $P(n)$ be “$n$ can be written as a product of primes”.

**Induction Hypothesis:** Assume that $P(j)$ is true for $1 < j < k$.

**Base Case:** We want to show that $P(2)$ is true. This is clearly true as 2 is a prime.

**Induction Step:** We want to show that $P(k+1)$ is true.

**Case I:** $k+1$ is prime. In this case we are done.

**Case II:** $k+1$ is composite. Then,

$$k + 1 = a \times b,$$

for some $a$ and $b$ s.t. $2 \leq a \leq b < k + 1$

By induction hypothesis, $a$ is a prime or it can be written as a product of primes. The same applies to $b$. Since $k + 1 = a \times b$, it can be written as a product of primes, namely those primes in the factorization of $a$ and those in the factorization of $b$. 


Example. Prove that, for any positive integer \( n \), if \( x_1, x_2, \ldots, x_n \) are \( n \) distinct real numbers, then no matter how the parenthesis are inserted into their product, the number of multiplications used to compute the product is \( n - 1 \).

Solution. Let \( P(n) \) be the property that “If \( x_1, x_2, \ldots, x_n \) are \( n \) distinct real numbers, then no matter how the parentheses are inserted into their product, the number of multiplications used to compute the product is \( n - 1 \)”.

Induction Hypothesis: Assume that \( P(j) \) is true for all \( j \) such that \( 1 \leq j \leq k \).

Base Case: \( P(1) \) is true, since \( x_1 \) is computed using 0 multiplications.

Induction Step: We want to prove \( P(k + 1) \). Consider the product of \( k + 1 \) distinct factors, \( x_1, x_2, \ldots, x_{k+1} \). When parentheses are inserted in order to compute the product of factors, some multiplication must be the final one. Consider the two terms, of this final multiplication. Each one is a product of at most \( k \) factors. Suppose the first and the second term in the final multiplication contain \( f_k \) and \( s_k \) factors. Clearly, \( 1 \leq f_k, s_k \leq k \). Thus, by induction hypothesis, the number of multiplications to obtain the first term of the final multiplication is \( f_k - 1 \) and the number of multiplications to obtain the second term of the final multiplication is \( s_k - 1 \). It follows that the number of multiplications to compute the product of \( x_1, x_2, \ldots, x_k, x_{k+1} \) is

\[
(f_k - 1) + (s_k - 1) + 1 = f_k + s_k - 1 = k + 1 - 1 = k
\]

Example. The game of NIM is played as follows: Some positive number of sticks are placed on the ground. Two players take turns, removing one, two or three sticks. The player to remove the last stick loses.

A winning strategy is a rule for how many sticks to remove when there are \( n \) left. Prove that the first player has a winning strategy iff the number of sticks, \( n \), is not \( 4k + 1 \) for any \( k \in \mathbb{N} \).

Solution. We will show that if \( n = 4k + 1 \) then player 2 has a strategy that will force a win for him, otherwise, player 1 has a strategy that will force a win for him.

Let \( P(n) \) be the property that if \( n = 4k + 1 \) for some \( k \in \mathbb{N} \) then the first player loses, and if \( n = 4k, 4k + 2, \text{or } 4k + 3 \), the first player wins. This exhausts all possible cases for \( n \).

Induction Hypothesis: Assume that for some \( z \geq 1 \), \( P(j) \) is true for all \( j \) such that \( 1 \leq j \leq z \).

Base Case: \( P(1) \) is true. The first player has no choice but to remove one stick and lose.

Induction Step: We want to prove \( P(z + 1) \). We consider the following four cases.

- Case I: \( z + 1 = 4k + 1 \), for some \( k \). We have already handled the base case, so we can assume that \( z + 1 \geq 5 \). Consider what the first player might do to win: he can remove 1, 2, or 3 sticks. If he removes one stick then the remaining number of sticks \( n = 4k \). By strong induction, the player who plays at this point has a winning strategy. So the player who played first loses. Similarly, if the first player removes two sticks or three sticks, the remaining number of sticks is \( 4(k - 1) + 3 \) and \( 4(k - 1) + 2 \) respectively. Again, the first player loses (using induction hypothesis). Thus, in this case, the first player loses regardless of what move he/she makes.

- Case II: \( z + 1 = 4k \), or \( z + 1 = 4k + 2 \), or \( z + 1 = 4k + 3 \). If the first player removes three
sticks in the first case, one stick in the second case, and two sticks in the third case then the second player sees $4(k - 1) + 1$ sticks in the first case and $4k + 1$ sticks in the other two cases. By induction hypothesis, in each case the second player loses.

**Example.** Prove that the two forms of induction, weak induction and strong induction, are equivalent. In other words, prove that any statement that admits a strong induction proof can be proved using weak induction and vice-versa.

**Solution.** Suppose we want to show that a $P(n)$ is true for all positive integers $n \geq n_0$. The two forms of inductive proofs are as follows.

**Weak Induction:** Assume that

(a) $P(n_0)$ is true
(b) For any $k \geq n_0$, $P(k) \implies P(k + 1)$ is true.

Then, $P(n)$ is true for all positive integers $n \geq n_0$.

**Strong Induction:** Assume that

(a) $P(n_0)$ is true
(b) For any $k \geq n_0$, $P(n_0) \land P(n_0 + 1) \land \cdots \land P(k) \implies P(k + 1)$ is true.

Then, $P(n)$ is true for all positive integers $n \geq n_0$.

We will show that it is always possible to convert a strong induction proof into a weak induction proof and vice-versa.

The conversion from a weak induction proof to a strong induction proof is trivial, since (b) implies (a).

We now show that a strong induction proof can be converted to a weak induction proof. Let

$$Q(n) \doteq P(n_0) \land P(n_0 + 1) \land \cdots \land P(n)$$

**Induction Hypothesis:** Assume that $Q(k)$ is true for some $k \geq n_0$.

**Base Case:** Since $Q(n_0) = P(n_0)$ and we know that $P(n_0)$ is true from (a), $Q(n_0)$ is true.

**Induction Step:** We want to show that $Q(k) \implies Q(k + 1)$. We have

\[
Q(k) \implies P(k + 1) \quad \text{(from (a))}
\]

\[
\therefore Q(k) \implies Q(k) \land P(k + 1)
\]

\[
\therefore Q(k) \implies Q(k + 1)
\]

Thus we have converted a strong induction proof in $P$ to a weak induction proof in $Q$. 