- Exam 1
  - next Thursday

- No class next Thursday

- OH TODAY
  - 1-2 pm ET.

Ex. k people

- n days in a year

Pr[ at least two people have the same birthday ] ?

What should be the value of k if we want this prob to be \( \geq \frac{1}{2} \).

\[ S_2 \subseteq \{ \underbrace{\{2, 3, 8, n\}}_{k-tuple} \ldots \} \]

Let \( B \) be the event that \( \geq 2 \) people
have the same birthday.

\[ \Pr[B] = 1 - \Pr[\bar{B}] \]

\[ = 1 - \frac{n \times (n-1) \times (n-2) \times \ldots \times n-(k-1)}{n^k} \]

\[ = 1 - \frac{P(n,k)}{n^k} \]

\[ 1 - \frac{P(n,k)}{n^k} \geq \frac{1}{2} \]

Support \( n = 365 \) then \( k = 23 \).

\( k = 60 \), \( \Pr[B] = 99.4\% \).

**Conditional Probability**

\[ \forall \omega \in B, \quad \Pr(\omega | B) = \frac{\Pr(\omega)}{\Pr(B)} \]
\[ \Pr(A \mid B) : \text{prob of event A given that event B has happened.} \]

From last week we know that for any event \( E \),

\[ \Pr(E) = \sum_{\omega \in E} \Pr(\omega) \]

Our event of interest = \( A \cap B \)

\[ \therefore \Pr(A \mid B) = \sum_{\omega \in A \cap B} \Pr(\omega \mid B) \]

\[ = \sum_{\omega \in A \cap B} \frac{\Pr(\omega)}{\Pr(B)} \]

\[ \Pr(A \mid B) = \frac{\Pr(A \cap B)}{\Pr(B)} \]

Ex: Two fair coins.

\[ \Pr(\text{both coins give HEADS given that} \]

...
one of the flips gives HEADS?)

Soh: \[ \frac{1}{2} \times \nabla \]

\[ \Pr(A | B) \]

\[ \Pr(w | B), \forall w \in B = \frac{1}{3}. \quad \nabla \Pr(w | B) = \frac{1}{4} \]

A: event that both flips give HEADS.

B: \[ \nabla \] one of the flips gives HEADS.

\[ \Pr(A | B) = \frac{\Pr(A \cap B)}{\Pr(B)} \]

\[ = \frac{\frac{1}{4}}{\frac{3}{4}} \]

\[ = \frac{1}{3}. \]

\[ \Pr(\text{both flips give HEADS given that the first flip gives HEADS}) = \frac{1}{3}. \times \]
C: event that the first flip gives HEADs.

\[
\Pr(A \mid C) = \frac{\Pr(A \cap C)}{\Pr(C)}
\]

\[
= \frac{\frac{1}{4}}{\frac{1}{2}} = \frac{1}{2}.
\]

Multiplication Rule:

\[
\Pr(A \cap B) = \Pr(A) \cdot \Pr(B \mid A).
\]

\[
\Pr(A \cap B \cap C) = \Pr(A) \cdot \Pr(B \mid A) \cdot \Pr(C \mid A \cap B).
\]

Ex: new car battery

- > 10k miles: 0.8

- > 20k miles: 0.4
battery is still working after 10k miles.

\[ \Pr[ \text{total life } > 20k \text{ miles}] \]

**Solu.** \( L_{10} \): event that the battery functions after 10k miles.

\[ L_{20} : \quad \text{after 20k miles.} \]

\[ \Pr[ L_{20} \cap L_{10} ] \quad \times \]

\[ \Pr[ L_{20} | L_{10} ] = \frac{\Pr[ L_{20} \cap L_{10} ]}{\Pr[ L_{10} ]} \]

\[ = \frac{0.4}{0.8} = \frac{1}{2} \]

**Ex.** Urn

- 5 white, 7 black
- Each time a ball is selected
- Color is noted
- Replaced in the urn along with
  two other balls of the same color.

- \( \Pr \left[ \text{first two balls are black and the}
\right.

\right. \text{next two are white] ?} \)

\[ \text{So: } B_1: \text{ event that 1st ball is black.} \]
\[ B_2: \text{ " " 2nd " " " } \]
\[ W_3: \text{ " " 3rd " white.} \]
\[ W_4: \text{ " " 4th " white.} \]

\( \Pr \left[ B_1 \cap B_2 \cap W_3 \cap W_4 \right], \Pr \left[ W_4 \mid W_3 \mid B_2 \mid B_1 \right], \)

\[ \Pr \left[ B_1 \cap B_2 \cap W_3 \cap W_4 \right] = \Pr \left[ B_1 \right] \cdot \Pr \left[ B_2 \mid B_1 \right] \cdot \Pr \left[ W_3 \mid B_2 \cap B_1 \right] \]
\[ \Pr \left( \left\{ \omega_1, \omega_2, \omega_3 \right\} \right) = \frac{7}{12} \cdot \frac{9}{14} \cdot \frac{5}{16} \cdot \frac{7}{18} \cdot \Pr \left( \omega_4 \mid B, \omega_2, \omega_3 \right) \]

**Total Probability Theorem**

\[ \Omega \]

\[ \Pr \left( E \right) = \sum_{\omega \in E} \Pr \left( \omega \right) \]

\[ = \sum_{\omega \in E \cap A_1} \Pr \left( \omega \right) + \sum_{\omega \in E \cap A_2} \Pr \left( \omega \right) + \sum_{\omega \in E \cap A_3} \Pr \left( \omega \right) + \sum_{\omega \in E \cap A_4} \Pr \left( \omega \right) \]

\[ = \Pr \left( E \cap A_1 \right) + \Pr \left( E \cap A_2 \right) + \Pr \left( E \cap A_3 \right) + \Pr \left( E \cap A_4 \right) \]
\[ \Pr(E \cap A_4) \]

\[ = \sum_i \Pr(E \cap A_i) \]

\[ = \sum_i \Pr(A_i) \cdot \Pr(E | A_i). \]

**Ex:** Medical test

**Affected person**
- True +ve : 95%  
  - False -ve : 5%

**Unaffected person**
- False +ve : 1%  
  - True -ve : 99%

0.5% of the population actually
Have the condition.

Prob. that the person has the condition given that the test is positive.

Soln:

P: event that the test is true.
C: "" the person has the condition.

$$\Pr(C \mid P) = \frac{\Pr(C \land P)}{\Pr(P)}$$

$$= \frac{0.95}{0.96}$$

$= 0.956$$
\[ P(C) = P(C \cap P) + P(C \cap \overline{P}) \]

\[ = P(C) \cdot P(P|C) + P(C) \cdot P(\overline{C}|P) \]

\[ = \left( 0.005 \right) \cdot \left( 0.95 \right) + \left( 0.995 \right) \cdot \left( 0.01 \right) \]

\[ = (0.005)(0.95) + (0.995)(0.01) \]

Our answer is:

\[ \frac{(0.005)(0.95)}{(0.005)(0.95) + (0.995)(0.01)} \]
\[ \text{Ex: Transmitter sends binary bits} \]

\[ - \quad 80\% \quad 0's \quad \quad \quad \quad \quad \]

\[ - \quad 20\% \quad 1's \quad \quad \quad \quad \]

\[ \text{when a 0 is sent} \]

\[ \text{receiver detects it correctly } 80\% \text{ time} \]

\[ \text{when a 1 is sent} \]

\[ \text{receiver detects it correctly } 90\% \text{ time} \]

\[ \text{Pr[1 is sent and a 1 is received]}? \]

\[ \text{Soln: } S_0 : \text{ event that a 0 is sent} \]

\[ S_1 : \quad \quad \quad \text{a 1 is sent} \]
\[ R_0 : \text{"0 received".} \]
\[ R_1 : \text{"1 received".} \]

\[ P_r \{ S_i \land R_i \} = P_r \{ S_i \} \cdot P_r \{ R_i | S_i \} \]
\[ = (0.2) \cdot (0.9) \]
\[ = 0.18. \]

(b) If a 1 is received, prob. that I was sent?

\[ P_r \{ R_i | S_i \} \times \]

\[ P_r \{ S_i | R_i \} = \frac{P_r \{ S_i \land R_i \}}{P_r \{ R_i \}}. \]

\[ = \frac{0.18}{P_r \{ R_i \land S_0 \} + P_r \{ R_i \land \neg S_i \}}. \]
0.18

\[ P(X = S_0) \cdot P(R_1 | S_0) + 0.18 \]

\[ 0.18 \]

\[ (0.8) \cdot (0.2) + 0.18 \]

\[ S2.9 \% \]