Recitation Guide - Week 2

Topics Covered: Sets, Counting, Proofs

Multiplication Rule: If a procedure can be broken down into \( k \) steps, where the \( i^{th} \) step can be done in \( n_i \) ways (where each \( n_i \) is independent of the preceding steps), the entire procedure can be done in \( n_1 \times n_2 \times n_3 \times \cdots \times n_k \) ways.

Problem 1:

1. List the members of the following sets:
   (a) \( \{ x \mid 2x \text{ is a positive integer less than 9} \} \)
   (b) \( \{ x \mid x \text{ is the cube of a natural number, and } x < 100 \} \)
   (c) \( \{ x \mid x \text{ is a positive integer such that } x^2 = 9 \text{ and } x < 3 \} \)

2. What is the cardinality of each of the following sets?
   (a) \( \{ \{ a \} \} \)
   (b) \( \{ a, \{ a \}, \emptyset \} \)
   (c) \( \{ a, \{ a \}, \{ a, b \} \} \)
   (d) \( 2^{\{a\}} \)

3. Determine whether each of the following is true or false:
   (a) \( \{x\} \subseteq \{x\} \)
   (b) \( \{x\} \subset \{x\} \)
   (c) \( x \in \{\{x\}\} \)

Solution:

1. (a) \( \left\{ \frac{1}{2}, 1, \frac{3}{2}, 2, \frac{5}{2}, 3, \frac{7}{2}, 4 \right\} \)
   (b) \( \{0, 1, 8, 27, 64\} \)
   (c) \( \emptyset \)

2. (a) 1
   (b) 3
   (c) 3
   (d) 2. The powerset of \( \{\{a\}\} \) is \( \{\{\{a\}\}, \emptyset\} \) since there is only one element in it.

3. (a) True
   (b) False
   (c) False. The actual element in the set is \( \{x\} \)
Problem 2: Show that for any two integers $m$ and $n$, $m^2 + n^2$ has the same parity as $m + n$.

Solution:

Consider the following two cases.

Case 1: $m + n$ is odd

We can write $m + n = 2k + 1$, for some $k \in \mathbb{Z}$. Then, we have,

$$m^2 + n^2 = (m + n)^2 - 2mn$$
$$= (2k + 1)^2 - 2mn$$
$$= 4k^2 + 4k + 1 - 2mn$$
$$= 2(2k^2 + 2k - mn) + 1$$

which is odd, as $2k^2 + 2k - mn \in \mathbb{Z}$.

Case 2: $m + n$ is even

We can write $m + n = 2k$, for some $k \in \mathbb{Z}$. Similarly, we have,

$$m^2 + n^2 = (m + n)^2 - 2mn$$
$$= (2k)^2 - 2mn$$
$$= 4k^2 - 2mn$$
$$= 2(2k^2 - mn)$$

which is even, as $2k^2 - mn \in \mathbb{Z}$.

We have shown that $m + n$ has the same parity as $m^2 + n^2$ and we are done.
Problem 3:

Your favorite pizza place in the world, Yonah’s Pizzeria, is known for its variety of different pizzas. Yonah’s has 5 different kinds of tomato sauces and 6 different kinds of cheese. In addition, you can add one of any 20 different toppings, which are optional. On top of all of these choices, you can choose a thin crust or a thick crust. How many different pizzas can you possibly order from Yonah’s?

Solution:

We will apply the Multiplication Rule:

Step 1: Choose the sauce. (5 ways)

Step 2: Choose the cheese. (6 ways)

Step 3: Choose the topping. (20 toppings + 1 option for no toppings = 21 ways)

Step 4: Choose the crust. (2 ways)

Multiplying all of these, we get $5 \times 6 \times 21 \times 2 = 1260$ pizzas.