CIS 160
Recitation Guide - Week 2

Topics Covered: Proof Techniques: Contra*, Truth Tables, Combinatorial Proofs, Multisets, Stars and Bars, Functions

Problem 1:
(a) Let \( a, b \in \mathbb{Z} \) and \( n \) is a positive integer. If \( n \) does not divide \( ab \) (we use the notation \( n \nmid ab \)), then \( n \) does not divide \( a \) and \( n \) does not divide \( b \) (\( n \nmid a \) and \( n \nmid b \)).

(b) Prove that \( \sqrt{6} \) is irrational.

Solution:
(a) We can prove the contrapositive of the above statement: if \( n \mid a \) or \( n \mid b \), then \( n \mid ab \). To prove this, assume \( n \mid a \) without loss of generality. Then \( a = kn \) for some \( k \in \mathbb{Z} \). Hence \( ab = knb = n \times kb \). Since \( k, b \in \mathbb{Z} \), we know that \( kb \) must be an integer, so \( n \mid ab \).

(b) We will use a proof by contradiction. Assume for the sake of contradiction that \( \sqrt{6} \) is rational. By the definition of a rational number, write \( \sqrt{6} \) as \( \frac{a}{b} \) where \( a \) and \( b \) have no common divisors other than 1 (we call them relatively prime or coprime natural numbers) and \( b \neq 0 \). This means that
\[
\frac{6}{b^2} = \frac{a^2}{b^2} = \frac{6b^2}{a^2}
\]
If \( 6 \mid a^2 \), then \( 2 \mid a^2 \) which implies that \( a \) must be even (recall Slide 12 of Lecture 4). Because \( a \) is even, let \( a = 2c \) for some integer \( c \).
\[
6b^2 = a^2
\]
\[
2 \times 3 \times b^2 = (2c)^2
\]
\[
2 \times 3 \times b^2 = 2 \times 2 \times c^2
\]
\[
3b^2 = 2c^2
\]
If \( 2 \mid (3b^2) \), then \( 2 \mid b^2 \) which implies that \( b \) must be even (see Lemma above). So, clearly, \( a \) and \( b \) are both even. However, this presents a contradiction: \( a \) and \( b \) must be relatively prime natural numbers, and thus cannot both be divisible by the same factor, 2.
Problem 2:

Show that \((p \Rightarrow q) \vee (\neg p \Rightarrow r)\) is not logically equivalent to \((p \Rightarrow q) \wedge (\neg p \Rightarrow r)\). Which one best captures “if \(p\) then \(q\) else \(r\)?

Solution:

Consider the following truth table:

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Since not all the truth assignments of \(p, q, r\) yield the same truth value of \((p \Rightarrow q) \vee (\neg p \Rightarrow r)\) and \((p \Rightarrow q) \wedge (\neg p \Rightarrow r)\), they are NOT logically equivalent. For example, when \(p = T, q = F, r = T\), we observe that \((p \Rightarrow q) \vee (\neg p \Rightarrow r) = T\) while \((p \Rightarrow q) \wedge (\neg p \Rightarrow r) = F\).

Now, which one best captures “if \(p\) then \(q\) else \(r\)?” We can interpret this statement as follows: whenever \(p\) is true, the truth value is determined by the truth value of \(q\) and whenever \(p\) is false, the truth value is determined by the truth value of \(r\).

If we take a look at the truth table above, we see that whenever \(p\) is true, the truth value of \((p \Rightarrow q) \wedge (\neg p \Rightarrow r)\) follows the truth value of \(q\) and whenever \(p\) is false, the truth value of \((p \Rightarrow q) \wedge (\neg p \Rightarrow r)\) follows the truth value of \(r\). However, such relationships do not hold with \((p \Rightarrow q) \vee (\neg p \Rightarrow r)\). Hence, \((p \Rightarrow q) \wedge (\neg p \Rightarrow r)\) best captures “if \(p\) then \(q\) else \(r\).”
Problem 3:
The janitor needed to distribute soap bars and toilet paper to customers of the hotel. He starts his shift with 10 bars of soap and 10 rolls of toilet paper. After the 6th room, he discovers that he has run out of supplies. Most importantly, he does not remember when his supplies ran out (meaning he could have used all his supplies in the first room). How many ways could he have distributed the toilet paper rolls and soap bars to the different rooms? He cannot tell the difference between any two toilet paper rolls and between any two soap bars. However, he can easily tell the difference between toilet paper and soap bars.

Solution:
We can break this problem down into separate stars and bars problems and combine them at the end.

There are 6 rooms in which we distribute 10 toilet paper rolls. Arrange 10 stars in a row. These stars represent the toilet paper rolls. Since the toilet paper rolls are indistinguishable, their ordering is irrelevant. We now also have $6 - 1 = 5$ bars to represent the rooms. Place the 5 bars between some of the stars (toilet paper rolls). The bars would then separate the stars into 6 parts, each of which represent one room. The number of arrangements of toilet paper rolls is then

$$\binom{10 + 6 - 1}{10} = \binom{15}{10}$$

We can do the same for the soap bars. 10 indistinguishable soap bars distributed along 6 rooms gives us

$$\binom{10 + 6 - 1}{10} = \binom{15}{10}$$

But this isn’t the end! We still have to combine them. We can do this by using the multiplication rule, since the order in which the toilet paper rolls is distributed is independent of the order in which the soap bars are distributed.

Step 1: Choose the order in which toilet paper rolls is distributed. From above, there are $\binom{15}{10}$ ways.

Step 2: Choose the order in which soap bars are distributed. Again, there are $\binom{15}{10}$ ways.

which gives us

$$\binom{15}{10} \times \binom{15}{10} = \binom{15}{10}^2$$
Problem 4: Give combinatorial proofs for the following:

(a) \( \binom{n}{k} = \binom{n-2}{k} + 2\binom{n-2}{k-1} + \binom{n-2}{k-2} \), where \( n \geq k \geq 2 \)

(b) \( \frac{n!}{i!j!k!} = \binom{n}{i} \times \binom{j+k}{k} \), where \( i, j, k \) are natural numbers and \( n = i + j + k \)

Solution:

(a) Consider the following counting problem:

Given a set of \( n - 2 \) distinct (non-captain) players and 2 distinct captains, Meri and Kaan, how many ways are there to form a team of \( k \) total players that have 0, 1, or 2 captains.

LHS: Since we can have any number of captains, we can treat the captains as part of the set of players and rewrite the question as “how many ways can we choose \( k \) people from a total of \( n \) people.” There are \( \binom{n}{k} \) ways to do so, which gives us the LHS.

RHS: Consider the three cases on the number of captains.

Case 1: We choose zero captains for the team. In this case, we can rewrite the question as “how many ways can we choose \( k \) people from a total of \( n - 2 \) people.” There are \( \binom{n-2}{k} \) ways to do so.

Case 2: We choose one captain for the team. In this case, we can use two steps to choose the captain and the team members.

   Step 1: Choose a captain to be on the team.
   Step 2: Choose the other \( k - 1 \) members on the team.

In Step 1, there are 2 captains and we can choose one of them, so there are \( \binom{2}{1} = 2 \) ways to do so. In Step 2, there are \( n - 2 \) players that we can choose and we have room for \( k - 1 \) of them, so there are \( \binom{n-2}{k-1} \) ways to do so. Hence, by the Multiplication Rule, we have \( 2\binom{n-2}{k-1} \) ways to form teams with one captain in them.

Case 3: We choose two captains for the team. In this case, we can use two steps to choose the captains and the team members.

   Step 1: Choose both captains to be on the team.
   Step 2: Choose the other \( k - 2 \) members on the team.

In Step 1, there are two captains to choose from and we choose both of them, so there are \( \binom{2}{2} = 1 \) way to do so. In Step 2, there are \( n - 2 \) players to choose from and we have room for \( k - 2 \) of them, so there are \( \binom{n-2}{k-2} \) ways to do so. Hence, by the Multiplication Rule, there are \( \binom{n-2}{k-2} \) ways to form teams with both captains in them.

Thus, by the Addition Rule, the total number of ways to form teams of \( k \) total players with 0, 1, or 2 captains is

\[ \binom{n-2}{k} + 2\binom{n-2}{k-1} + \binom{n-2}{k-2} \]

which give us the RHS.
(b) We can pose the question: Given $i$ apples, $j$ oranges, and $k$ lemons, how many ways can we arrange the $n$ fruits in a line? Fruits of the same type are indistinguishable from one another, but fruits of different types are distinguishable.

**LHS:** We can consider this as a problem involving permutations of a bag $\{i \cdot n_1, j \cdot n_2, k \cdot n_3\}$. We have a multiset so applying the formula for permutations of a multiset, the total number of permutations of this bag is just

$$\frac{n!}{i!j!k!}$$

**RHS:** We can also solve this by Multiplication Rule.

*Step 1:* Choose $i$ positions out of $n$ positions to place the apples. Since the apples are indistinguishable, there is one way to arrange these apples once the positions have been picked. $\binom{n}{i}$ ways

*Step 2:* Choose $k$ positions out of the remaining $n - i = j + k$ positions to place the lemons. Similar to the apples, since the lemons are indistinguishable, there is only one way to arrange these lemons. $\binom{j+k}{k}$ ways

*Step 3:* Choose $j$ positions out of the remaining $n - i - k = j$ positions to place the oranges. There is only one way to do this since the oranges are indistinguishable. (1 way)

Since the steps are independent, by the Multiplication Rule, there are

$$\binom{n}{i} \times \binom{j+k}{k}$$

ways of arranging the fruits in a line.

Since each counting procedure above answers the same question, we see that the two sides must be equal, as desired.
Problem 5: Let \( m, n \geq 2 \). Define:

\[
\begin{align*}
f : [1..m] \times [1..n] &\rightarrow [2..m+n] \\
&\text{by } f(x, y) = x + y.
\end{align*}
\]

Is \( f \) an injection? Is \( f \) a surjection? Prove your answers.

Solution:

\( f \) is not injective. In order to show this, we will provide two elements \((a, b), (c, d) \in [1..m] \times [1..n]\) such that \((a, b) \neq (c, d)\) but \(f(a, b) = f(c, d)\). To this end, consider \((1, 2)\) and \((2, 1)\). Clearly, \((1, 2) \neq (2, 1)\). However, since \(f(1, 2) = 3 = f(2, 1)\), they both map onto the same element in the codomain, and we conclude that \( f \) cannot be injective.

\( f \) is surjective. To prove surjectivity, we must show that \( \forall z \in [2..m+n], \exists x \in [1..m], y \in [1..n] \) such that \( f(x, y) = z \). Intuitively, we want to show that everything in the codomain of \( f \) is mapped to by at least one element from the domain. Consider any \( z \) in the codomain, i.e., any \( z \in [2..m+n] \).

We consider the following cases:

Case 1: \( 2 \leq z \leq m + 1 \).

In this case, let \( y = 1 \). Then we need \( x = z - 1 \), by definition of \( f \). Substituting into the fact that \( 2 \leq z \leq m + 1 \), we see that \( 1 \leq x \leq m \), meaning \( x \) will always be an element of \([1..m]\). Thus, we know that \((x, 1) \in [1..m] \times [1..n]\), and we have found an element of the domain, namely \((x, 1)\), such that \( f(x, 1) = z \).

Case 2: \( m + 1 \leq z \leq m + n \).

In this case, let \( x = m \). Then we need \( y = z - m \), by definition of \( f \). As above, by substituting into the fact that \( m + 1 < z \leq m + n \), we see that \( 1 \leq y \leq n \), meaning \( y \) will always be an element of \([1..n]\). As above, we know that \((m, y) \in [1..m] \times [1..n]\), and we have found an element of the domain, namely \((m, y)\), such that \( f(m, y) = z \).

In both cases, we have found an element of the domain \((x, y)\) such that \( f(x, y) = z \), so we know that \( f \) is surjective.
**Problem 6:** Let \( n \geq 1 \). Define \( f : 2^{[1..n]} \to \mathbb{N} \) by \( f(S) = |2^S| \). What is \(|\text{Ran}(f)|\)?

**Solution:**

Let \( A = [1..n] \). Note that the domain of the function \( f \) is the powerset of \( A \). Let \( S \) be a subset of \( A \). Then, \( \text{Ran}(f) = \{ x \mid \exists S \subseteq A \ (|2^S| = x) \} \). We know that \(|2^S| = |2|^{|S|} \) from lecture, which implies that \( \text{Ran}(f) = \{ x \mid \exists S \subseteq A \ (|2|^{|S|} = 2^{|S|} = x) \} \). In other words, \( \text{Ran}(f) \) consists of all the possible values of \( 2^{|S|} \).

Let \( X \) be all the possible values of \(|S|\). Let \( Y \) be all the possible values of \( 2^{|S|} \).

**Claim:** \(|X| = |Y|\).

**Proof:** It suffices to construct a bijective map \( f \) between \( X \) and \( Y \), as by the bijection rule, if \( f : X \to Y \) is a bijection, then \(|X| = |Y|\).

Define \( f : X \to Y \) by \( f(a) = 2^a \).

First, we show that \( f \) is injective. Consider arbitrary elements \( a, b \in X \) such that \( f(a) = f(b) \). Since \( f(a) = 2^a = 2^b = f(b) \), we have \( 2^a = 2^b \). Since \(|S| \geq 0 \), we know that \( a, b \geq 0 \), which means that \( 2^a = 2^b \implies a = b \). Hence, \( f \) is injective.

Now, we show that \( f \) is surjective. Consider an arbitrary element \( c \in Y \). We will show that there exists \( x = \log_2 c \in X \) such that \( f(x) = c \). We need to check that (1) \( x \in X \) and (2) \( f(x) = c \).

1. \( 2^0 \leq c \leq 2^n \implies 0 \leq \log_2 c \leq n \implies 0 \leq x \leq n \implies x \in X \)
2. \( f(x) = f(\log_2 c) = 2^{\log_2 c} = c \)

Hence, \( f \) is surjective. Since \( f \) is both injective and surjective, \( f \) is bijective, which implies that \(|A| = |B| \) by the bijection rule.

So, to compute \(|\text{Ran}(f)|\), it suffices to count the number of distinct values of \(|S|\). Observe that each element in \( A \) can either be included or not when we construct a subset, which means that the possible values of \(|S|\) are 0, 1, \ldots, \( n \). Since there are \( n + 1 \) distinct possible values that \(|S|\), thereby \( 2^{|S|} = |2|^{|S|} = 2^5 \) can be, \(|\text{Ran}(f)| = n + 1 \).