Recitation Guide - Week 4

Topics Covered: Stars and Bars, Truth Tables, Combinatorial Proofs

Problem 1: The janitor needed to distribute soap bars and toilet paper to customers of the hotel. He starts his shift with 10 bars of soap and 10 rolls of toilet paper. After the 6th room, he discovers that he has run out of supplies. Most importantly, he does not remember when his supplies ran out (meaning he could have used all his supplies in the first room). How many ways could he have distributed the toilet paper rolls and soap bars to the different rooms? He cannot tell the difference between any two toilet paper rolls and between any two soap bars. However, he can easily tell the difference between toilet paper and soap bars.

Solution:

We can break this problem down into separate stars and bars problems and combine them at the end.

There are 6 rooms in which we distribute 10 toilet paper rolls. Arrange 10 stars in a row. These stars represent the toilet paper rolls. Since the toilet paper rolls are indistinguishable, their ordering is irrelevant. We now also have $6 - 1 = 5$ bars to represent the rooms. Place the 5 bars between some of the stars (toilet paper rolls). The bars would then separate the stars into 6 parts, each of which represent one room. The number of arrangements of toilet paper rolls is then

$$\binom{10 + 6 - 1}{10} = \binom{15}{10}$$

We can do the same for the soap bars. 10 indistinguishable soap bars distributed along 6 rooms gives us

$$\binom{10 + 6 - 1}{10} = \binom{15}{10}$$

But this isn’t the end! We still have to combine them. We can do this by using the multiplication rule, since the order in which the toilet paper rolls is distributed is independent of the order in which the soap bars are distributed.

Step 1: Choose the order in which toilet paper rolls is distributed. From above, there are $\binom{15}{10}$ ways.

Step 2: Choose the order in which soap bars are distributed. Again, there are $\binom{15}{10}$ ways.

which gives us

$$\binom{15}{10} \times \binom{15}{10} = \binom{15}{10}^2$$
**Problem 2:**

Show that \((p \Rightarrow q) \lor (\neg p \Rightarrow r)\) is not logically equivalent to \((p \Rightarrow q) \land (\neg p \Rightarrow r)\). Which one best captures “if \(p\) then \(q\) else \(r\)?

**Solution:**

Consider the following truth table:

<table>
<thead>
<tr>
<th>(p)</th>
<th>(q)</th>
<th>(r)</th>
<th>(\neg p)</th>
<th>(p \Rightarrow q)</th>
<th>(\neg p \Rightarrow r)</th>
<th>((p \Rightarrow q) \lor (\neg p \Rightarrow r))</th>
<th>((p \Rightarrow q) \land (\neg p \Rightarrow r))</th>
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</thead>
<tbody>
<tr>
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Since not all the truth assignments of \(p, q, r\) yield the same truth value of \((p \Rightarrow q) \lor (\neg p \Rightarrow r)\) and \((p \Rightarrow q) \land (\neg p \Rightarrow r)\), they are NOT logically equivalent. For example, when \(p = T, q = F, r = T\), we observe that \((p \Rightarrow q) \lor (\neg p \Rightarrow r) = T\) while \((p \Rightarrow q) \land (\neg p \Rightarrow r) = F\).

Now, which one best captures “if \(p\) then \(q\) else \(r\)?” We can interpret this statement as follows: whenever \(p\) is true, the truth value is determined by the truth value of \(q\) and whenever \(p\) is false, the truth value is determined by the truth value of \(r\).

If we take a look at the truth table above, we see that whenever \(p\) is true, the truth value of \((p \Rightarrow q) \land (\neg p \Rightarrow r)\) follows the truth value of \(q\) and whenever \(p\) is false, the truth value of \((p \Rightarrow q) \land (\neg p \Rightarrow r)\) follows the truth value of \(r\). However, such relationships do not hold with \((p \Rightarrow q) \lor (\neg p \Rightarrow r)\). Hence, \((p \Rightarrow q) \land (\neg p \Rightarrow r)\) best captures “if \(p\) then \(q\) else \(r\).”
Problem 3: Give a combinatorial proof for the following, where \( n \geq k \geq 2 \):

\[
\binom{n}{k} = \binom{n-2}{k} + 2\binom{n-2}{k-1} + \binom{n-2}{k-2}
\]

Solution:

Consider the following counting problem:

Given a set of \( n - 2 \) distinct (non-captain) players and 2 distinct captains, Steph and Hannah, how many ways are there to form a team of \( k \) total players that have 0, 1, or 2 captains.

LHS: Since we can have any number of captains, we can treat the captains as part of the set of players and rewrite the question as “how many ways can we choose \( k \) people from a total of \( n \) people.” There are \( \binom{n}{k} \) ways to do so, which gives us the LHS.

RHS: Consider the three cases on the number of captains.

Case 1: We choose zero captains for the team. In this case, we can rewrite the question as “how many ways can we choose \( k \) people from a total of \( n - 2 \) people.” There are \( \binom{n-2}{k} \) ways to do so.

Case 2: We choose one captain for the team. In this case, we can use two steps to choose the captain and the team members.

Step 1: Choose a captain to be on the team.
Step 2: Choose the other \( k - 1 \) members on the team.

In Step 1, there are 2 captains and we can choose one of them, so there are \( \binom{2}{1} = 2 \) ways to do so. In Step 2, there are \( n - 2 \) players that we can choose and we have room for \( k - 1 \) of them, so there are \( \binom{n-2}{k-1} \) ways to do so. Hence, by the Multiplication Rule, we have \( 2\binom{n-2}{k-1} \) ways to form teams with one captain in them.

Case 3: We choose two captains for the team. In this case, we can use two steps to choose the captains and the team members.

Step 1: Choose both captains to be on the team.
Step 2: Choose the other \( k - 2 \) members on the team.

In Step 1, there are two captains to choose from and we choose both of them, so there are \( \binom{2}{2} = 1 \) way to do so. In Step 2, there are \( n - 2 \) players to choose from and we have room for \( k - 2 \) of them, so there are \( \binom{n-2}{k-2} \) ways to do so. Hence, by the Multiplication Rule, there are \( \binom{n-2}{k-2} \) ways to form teams with both captains in them.

Thus, by the Addition Rule, the total number of ways to form teams of \( k \) total players with 0, 1, or 2 captains is

\[
\binom{n-2}{k} + 2\binom{n-2}{k-1} + \binom{n-2}{k-2}
\]

which give us the RHS.