Recitation Guide - Week 3

Topics Covered: PIE, PHP, Induction, Probability

Problem 1: Prove using induction that for any positive integer \( n \) and for any \( d_0, d_1, \ldots, d_{n-1} \in [0..9] \) we have

\[
\sum_{j=0}^{n-1} d_j \cdot 10^j < 10^n
\]

Solution:

(BASE CASE) \( n = 1 \).

\[
\sum_{j=0}^{0} d_j \cdot 10^j = d_0 \cdot 10^0 = d_0
\]

\( d_0 \) can only take values from 0 through 9 inclusive, thus \( d_0 < 10^1 = 10 \). ✓

(INDUCTION STEP) Let \( k \in \mathbb{Z}^+ \) be arbitrary. Assume that \( \text{(IH)} \)

\[
\sum_{j=0}^{k-1} d_j \cdot 10^j < 10^k
\]

We now want to show

\[
\sum_{j=0}^{k} d_j \cdot 10^j < 10^{k+1}
\]

We see that we can show this as follows:

\[
\sum_{j=0}^{k} d_j \cdot 10^j = d_k \cdot 10^k + \sum_{j=0}^{k-1} d_j \cdot 10^j \quad \text{(splitting the sum)}
\]

\[
< d_k \cdot 10^k + 10^k \quad \text{(by IH)}
\]

\[
= (d_k + 1) \cdot 10^k
\]

\[
\leq (9 + 1) \cdot 10^k
\]

\[
= 10^{k+1}
\]

thus concluding our Induction Step and our proof.
**Problem 2:** Prove by induction that the number of diagonals in a convex polygon is \( \frac{n(n-3)}{2} \), where \( n \) is the number of sides of the polygon.

**Solution:**

We prove by induction on \( n \) that the number of diagonals in a convex polygon is \( \frac{n(n-3)}{2} \) where \( n \) is the number of sides of the polygon.

Notice that the polygon with the least number of sides is a triangle.

**(BASE CASE)** \( n = 3 \). The number of diagonals in a triangle is \( \frac{n(n-3)}{2} = \frac{3(3-3)}{2} = 0. \checkmark\)

**(INDUCTION STEP)** Let \( k \in \mathbb{N}, k \geq 3 \) be arbitrary. Assume that (IH) any convex polygon with \( k \) sides has \( \frac{k(k-3)}{2} \) diagonals. We want to show that any convex polygon with \( k + 1 \) sides has \( \frac{(k+1)(k-2)}{2} \) diagonals.

Consider an arbitrary convex polygon with \( k + 1 \) sides and label its vertices in clockwise order. If we draw a diagonal from vertex 1 to vertex 3, the resulting partitioned polygon has \( k \) sides. Then, by IH, the partitioned part of the polygon has \( \frac{k(k-3)}{2} \) diagonals. The number of diagonals from the missing vertex 2 is just \( k - 2 \), since it has a diagonal to every vertex other than 1 and 3. The edge we drew from vertex 1 to vertex 3 is also a diagonal in the original polygon. Then, the total number of diagonals in this polygon would be

\[
\begin{align*}
\frac{k(k - 3)}{2} + k - 2 + 1 &= \frac{k^2 - 3k + 2k - 4 + 2}{2} \\
&= \frac{k^2 - k - 2}{2} \\
&= \frac{(k + 1)(k - 2)}{2}
\end{align*}
\]

thus concluding our Induction Step and the proof.
**Problem 3:** Alice and Bob are playing a game in which there are two bags with an equal number of marbles in them. In this game, the two players take turns removing marbles from one of the bags. In each turn, the player can remove any positive number of marbles as long as they are all from the same bag. The winner of the game is the player that removes the last marble. In Alice and Bob’s current configuration, both bags initially start with the same number of marbles. Prove that one of them can guarantee a win.

**Solution:**

Consider the following strategy: the player who goes second always removes the same number of marbles as the player who went first, but from the other bag. If Bob goes first, Alice can always win by using this strategy.

Define $P(k)$ to be the claim that this strategy always works for bags that start with $k$ marbles each.

**Base Case:** $k = 1$. Bob goes first. His only move is to remove one marble from a bag. Alice then removes the last marble from the other bag. Thus the strategy works.

**Induction Step:** Assume $P(\ell)$ is true, for $1 \leq \ell \leq k$, for an arbitrary $k \in \mathbb{Z}$, $k \geq 1$. (IH)

We want to show that the claim still holds if each bag has $k + 1$ marbles. So, we start with two bags containing $k + 1$ marbles each. In Bob’s first move, he can remove $j$ number of marbles for $j \in \mathbb{Z}$, $1 \leq j \leq k + 1$.

**Case 1:** $j = k + 1$ (i.e. Bob removes all the marbles from a bag).

In this case, Alice can just take the $k + 1$ marbles in the other bag. Because she took the last marble, she wins.

**Case 2:** $1 \leq j \leq k$:

Thus, after the first move, the bags contain $k + 1 - j$ and $k + 1$ marbles. According to the strategy, Alice removes $j$ marbles from the other bag so that both bags now contain $k + 1 - j$ marbles. We can view the current state of the game as a new game in which both piles contain $k + 1 - j$ marbles. Since $1 \leq k + 1 - j \leq k$, we can apply the induction hypothesis to state that this strategy will always work.
Problem 4: A chess master who has 11 weeks to prepare for a tournament decides to play at least one game every day but, in order not to tire himself, he decides not to play more than 12 games during any calendar week. Show that there exists consecutive days during which the chess master will have played exactly 21 games.

Solution:
Let $a_i$, $1 \leq i \leq 77$, be the total number of games that the chess master has played during the first $i$ days. Note that the sequence of numbers $a_1, a_2, \ldots, a_{77}$ is a strictly increasing sequence. We have

$$1 \leq a_1 < a_2 < \ldots < a_{77} \leq 11 \times 12 = 132$$

Now consider the sequence $a_1 + 21, a_2 + 21, \ldots, a_{77} + 21$. We have

$$22 \leq a_1 + 21 < a_2 + 21 < \ldots < a_{77} + 21 \leq 153$$

Clearly, this sequence is also a strictly increasing sequence. The numbers $a_1, a_2, \ldots, a_{77}, a_1 + 21, a_2 + 21, \ldots, a_{77} + 21$ (154 in all) belong to the set $\{1, 2, \ldots, 153\}$. By the pigeonhole principle there must be two numbers out of the 154 numbers that must be the same. Since no two numbers in $a_1, a_2, \ldots, a_{77}$ are equal and no two numbers in $a_1 + 21, a_2 + 21, \ldots, a_{77} + 21$ are equal there must exist $i$ and $j$ such that $a_i = a_j + 21$. Hence during the days $j + 1, j + 2, \ldots, i$, exactly 21 games must have been played.
Problem 5:
Compute the probability of the event “when we roll \( n \) (distinguishable) fair dice any \( k \) of the dice show the same number while the other \( n - k \) show numbers different from the one shown by the \( k \) dice”. Assume \( n \geq 3 \) and \( \frac{n}{2} < k < n \).

Solution:
As discussed in class, we have a uniform probability space whose outcomes are sequences of length \( n \) of numbers from \([1..6]\). In other words, the \( \Omega \) is given by the cartesian product of \([1..6]\) \( \times \cdots \times [1..6] \) \( (n \text{ times}) \), i.e., \( \Omega = [1..6]^n \). By the Multiplication Rule, there are \( 6 \times \cdots \times 6 = 6^n \) such sequences so each outcome has probability \( \frac{1}{6^n} \).

Let \( E \) be the event where exactly \( k \) of the dice show the same number. We see that we are trying to find \( \Pr[E] \). To compute the desired probability, it suffices to count the cardinality of \( E \), i.e., the number of sequences (of interest) in which \( k \) positions have the same number from \( t \in [1..6] \) while the other \( n - k \) position show numbers different from \( t \). Such a sequence can be constructed as follows:

Step 1: Choose \( t \in [1..6] \). This can be done in 6 ways.
Step 2: Choose \( k \) of the \( n \) positions in the sequence. This can be done in \( \binom{n}{k} \) ways.
Step 3: Place \( t \) in each of these positions. This can be done in 1 way.
Step 4: For each of the remaining \( n - k \) positions choose a number from \([1..6] \setminus \{t\} \). We see that there are 5 such numbers, and \( n - k \) positions that we have left to fill. Thus, this can be done in \( 5^{n-k} \) ways.

By the Multiplication Rule, the number of sequences of interest is \( 6 \binom{n}{k} 5^{n-k} \). Hence, the probability we are asked for is given by:

\[
\Pr[E] = \frac{|E|}{|\Omega|} = \left( 6 \binom{n}{k} 5^{n-k} \right) \frac{1}{6^n} = \left( \binom{n}{k} \right) 5^{n-k} 6^{n-1}
\]

Aside:
Why doesn’t our method overcount? For example, let us consider the outcome \( T = (1,1,2,2,3,3) \in \Omega \) for \( n = 6 \). We could derive \( T \) by first picking \( t = 1 \) and assigning it to the first two dice. Then in step 4 we could generate \( (2,2,3,3) \). Similarly our method allows us to pick \( t = 3 \), assign it to the last two dice, and place \( (1,1,2,2) \) in the first four dice.

To see why this cannot happen look at the assumption that \( \frac{n}{2} < k < n \). We will now show that each event will only be counted once in our counting procedure. Let us consider any tuple \( T \) of length \( n \) which is in \( \Omega \). By our assumption more than half of these positions is filled by a number between 1 and 6. WLOG assume that this number is 1. We will show that it is impossible to construct this tuple again. First, the remaining \( n - k \) positions of this tuple will be uniquely generated by step 4 so we don’t have to worry about step 4 causing a repetition of this tuple.

Now let us consider choosing a different \( t \neq 1 \). We will show it is impossible to construct our tuple \( T \) again when \( t \neq 1 \). To see this, notice that because \( k > \frac{n}{2} \) any tuple constructed with \( t \) chosen in step 1 will have more than half of the positions will be filled with \( t \). But our tuple \( T \) had 1 in the more than half of its positions! So no matter what we do in steps 2-4 we could have never constructed \( T \) again.
Problem 6:
We have three wooden buckets, $T_A, T_B, T_C$ and we throw $n \geq 3$ metal keys in them. The key throws are mutually independent and each key is equally likely to land in each of the three buckets.

(a) Let $A$ be the event that after all keys are thrown bucket $T_A$ has at least one key in it and similarly associate an event $B$ with $T_B$. Are $A$ and $B$ independent? Justify your answer.

(b) Compute the probability that after all keys are thrown, each of the three buckets has at least one key in it. Justify your answer.

We define the sample space $\Omega$ to be the set of all ordered $n$-tuples such that each element is either $A, B$ or $C$, and represents the bucket each key goes into.

(a) For $i = 1, \ldots, n$ let $A_i$ be the event that key $i$ is thrown in bucket $T_A$. We have $\Pr[A_i] = \frac{1}{3}$. Clearly $A = A_1 \cup \cdots \cup A_n$ and since the events $A_1, \ldots, A_n$ are mutually independent we can compute complementarily:

$$\Pr[A] = \Pr[A_1 \cup \cdots \cup A_n] = 1 - \Pr[\overline{A}_1 \cap \cdots \cap \overline{A}_n]$$

$$= 1 - \prod_{i=1}^n (1 - \Pr[A_i])$$

$$= 1 - \left(1 - \frac{1}{3}\right)^n$$

$$= 1 - \left(\frac{2}{3}\right)^n$$

Similarly, $\Pr[B] = 1 - \left(\frac{2}{3}\right)^n$. To check independence we also need $\Pr[A \cap B]$.

Upon reflection, we notice that there is one aspect of the problem that we have not used yet: the keys get thrown only in $T_A, T_B$ and $T_C$. Thus, $\overline{A} \cap \overline{B}$, which means that both $T_A$ and $T_B$ are empty after all keys are thrown, is the same as the event “all keys get thrown in $T_C$” and therefore, by mutual independence, has probability $\left(\frac{1}{3}\right)^n$, as each key has a $\frac{1}{3}$ probability of being thrown into $T_C$. Now we can compute, using properties of probability and De Morgan’s Laws:

$$\Pr[A \cap B] = \Pr[A] + \Pr[B] - \Pr[A \cup B]$$

$$= \Pr[A] + \Pr[B] - (1 - \Pr[\overline{A} \cap \overline{B}])$$

$$= 1 - \left(\frac{2}{3}\right)^n + 1 - \left(\frac{2}{3}\right)^n - \left(1 - \left(\frac{1}{3}\right)^n\right)$$

$$= 1 - 2 \left(\frac{2}{3}\right)^n + \left(\frac{1}{3}\right)^n$$

But we also know that:

$$\Pr[A] \cdot \Pr[B] = \left(1 - \left(\frac{2}{3}\right)^n\right)\left(1 - \left(\frac{2}{3}\right)^n\right) = 1 - 2 \left(\frac{2}{3}\right)^n + \left(\frac{4}{9}\right)^n$$

Since $\frac{1}{3} \neq \frac{4}{9}$ it follows that $\Pr[A] \cdot \Pr[B] \neq \Pr[A \cap B]$ hence $A$ and $B$ are not independent.
(b) We continue with the notation introduced in part (a) and we also define $C$ to be the event “$T_C$ is not empty after all keys are thrown.” This part asks for $P(A \cap B \cap C)$. We are tempted to multiply probabilities but we do not know if $A, B, C$ are mutually independent. In fact, in part (a) we saw that $A \not\perp B$. Although it is still possible that $P(A \cap B \cap C) = P[A] \cdot P[B] \cdot P[C]$ there is no reason to hope for this here (and in fact we shall see that it does not hold).

Instead, we will use the Principle of Inclusion-Exclusion for three events:

$$P(A \cup B \cup C) = P[A] + P[B] + P[C] - P[A \cap B] - P[B \cap C] - P[C \cap A] + P[A \cap B \cap C]$$

Since we have at least one key, at least one of the buckets ends up non-empty. Hence $A \cup B \cup C = \Omega$, meaning $A \cup B \cup C$ consists of all the outcomes and has probability 1. From part (a) we have:

$$P[A] = P[B] = P[C] = 1 - \left(\frac{2}{3}\right)^n$$

$$P[A \cap B] = P[B \cap C] = P[C \cap A] = 1 - 2 \left(\frac{2}{3}\right)^n + \left(\frac{1}{3}\right)^n$$

We plug in and obtain

$$P[A \cap B \cap C] = P[A \cup B \cup C] - P[A] - P[B] - P[C] + P[A \cap B] + P[B \cap C] + P[C \cap A]$$

$$= 1 - 3 \left(1 - \left(\frac{2}{3}\right)^n\right) + 3 \left(1 - 2 \left(\frac{2}{3}\right)^n + \left(\frac{1}{3}\right)^n\right)$$

$$= 1 - 3 \left(\frac{2}{3}\right)^n + 3 \left(\frac{1}{3}\right)^n$$

Alternate Solution

We define the sample space $\Omega$ to be the set of all ordered $n$-tuples such that each element is either $A, B$ or $C$, and represents the bucket each key goes into. Notice that since each key thrown is equally likely to land in each of the three buckets, and are mutually independent, the sample space is uniform.

We define event $E$ to be the event where each bucket has at least one key in it. Since the sample space is uniform, we must have

$$P[E] = \frac{|E|}{|\Omega|}$$

We know that $|\Omega| = 3^n$ as there are 3 buckets to throw each key in. What remains is to count the number of ways to distribute the keys into buckets such that each bucket has at least one key. This is a familiar problem!

We count $|E|$ complementarily. Note that $A, B, C$ are sets that represent the number of ways to throw each key into three buckets such that bucket $T_A$ is empty, $T_B$ is empty or $T_C$ is empty respectively.

$$|E| = |A \cap B \cap C|$$

$$= |\Omega| - |\overline{A} \cup \overline{B} \cup \overline{C}|$$
By PIE, we have $|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|$. Furthermore,

$$
|A| = |B| = |C| = 2^n \\
|A \cap B| = |A \cap C| = |B \cap C| = 1 \\
|A \cap B \cap C| = 0
$$

Hence, we have that

$$
|E| = 3^n - |A \cup B \cup C| \\
= 3^n - (3 \cdot 2^n - 3)
$$

Finally,

$$
\Pr[E] = \frac{|E|}{\Omega} \\
= \frac{3^n - (3 \cdot 2^n - 3)}{3^n} \\
= 1 - 3 \left(\frac{2}{3}\right)^n + 3 \left(\frac{1}{3}\right)^n
$$