Problem 1: 
Prove that $G$ or the complement of $G$ is connected.

Solution:

If $G$ is connected we are done.

If $G$ is not connected then $G$ is composed of multiple connected components. We want to prove that given two arbitrary vertices in $G$ there must be a path between them in $\overline{G}$. Let these two arbitrary vertices be $u$ and $v$.

Case 1: $u$ and $v$ do not share an edge in $G$

This means they must share an edge in $\overline{G}$ and thus there is a path from $u$ to $v$ in $\overline{G}$.

Case 2: $u$ and $v$ share an edge in $G$

This means they were part of the same connected component in $G$. Take an arbitrary vertex $x$ in a different connected component in $G$. Edges $u - x$ and $v - x$ must both exist in $\overline{G}$. Thus, there is a path $u - x - v$ between vertices $u$ and $v$.

Thus, we have shown that there exists a path between any two arbitrary vertices in $\overline{G}$. By definition $\overline{G}$ must be connected. The claim is proved.
Problem 2: Prove that a graph $G = (V, E)$ is connected iff for every partition of $V$ into two disjoint, non-empty sets $S$ and $T$, there exists an edge between some vertex in $S$ and some vertex in $T$.

Solution:

($\Rightarrow$): We first show that if a graph $G = (V, E)$ is connected, then for every partition of $V$ into two disjoint, non-empty sets $S$ and $T$, there exists an edge between some vertex in $S$ and some vertex in $T$. Consider an arbitrary partition $V = S \cup T$ into two disjoint, non-empty sets $S$ and $T$. Let $x \in S$ and $y \in T$; since $G$ is connected, there must be a path $x \leadsto y$, say:

$$P = x - v_1 - v_2 - \ldots - v_{k-1} - y$$

We claim that there must be some edge from $S$ to $T$ in this path. Suppose towards contradiction that all edges are between two vertices in $S$ or two vertices in $T$. Since $x \in S$, we must have $v_1 \in S$. Similarly, we must then have $v_2 \in S$. We may continue this process to show that $v_{k-1} \in S$ (see if you can formally prove this with induction!), and $y \in S$, a contradiction.

($\Leftarrow$): We now show that, given a graph $G = (V, E)$, if for every partition of $V$ into two disjoint, non-empty sets $S$ and $T$, there exists an edge between some vertex in $S$ and some vertex in $T$, then $G$ is connected. We proceed by proving the contrapositive, namely, that if $G$ is not connected, then there exists a partition of $V$ into disjoint nonempty sets $S$ and $T$ with no edges between the two.

Since $G$ is not connected, it must have at least two connected components. Let $S$ be a connected component of $G$ and let $T = V \setminus S$. By definition of connected component, there is no edge from a vertex of $S$ to one in $T$ (if there were, we would violate the maximality condition). This gives us our desired partition.
Problem 3: Let $T$ be a tree where the maximum degree is $\Delta$. Prove that $T$ has at least $\Delta$ leaves.

Solution:

We will use the (non-standard) notation $\lambda(T)$ to denote the number of leaves in a tree $T$. Thus, we can rewrite the claim as $\lambda(T) \geq \Delta(T)$.

Direct Proof:

Let $v \in V$ have degree $\Delta$ in $T = (V, E)$. Consider the subgraph induced on the vertices $V \setminus \{v\}$. Each neighbor of $v$ is in a distinct component in this graph, because we have destroyed the unique path between any two of $v$'s neighbors in $T$. Thus there are $\Delta$ components, each of which is a tree.

There are two possibilities for each component. If a component is a single node, then this single node is a leaf adjacent to $v$ in $T$. If the component has at least 2 nodes, then it has at least 2 leaves. One of the leaves may be adjacent to $v$ and not a leaf in $T$. But the other leaf in this component is still a leaf in $T$. In any case, each component contains at least one leaf of $T$ and hence $T$ must have $\Delta$ leaves.

Maximal Path:

Let $v \in V$ have degree $\Delta$. For each $u_i, u_j \in N(v)$, let $P_{i,j}$ be a maximal path including $u_i - v - u_j$. Note that there must be at least $\binom{\Delta}{2}$ such paths, since any pair of starting edges gives a different path. We know that any such path $P_{i,j}$ must terminate in two leaves (call them $w_{i,j}$ and $x_{i,j}$).

Lastly, note that since there is a unique path between any two vertices in a tree, every pair of leaves admits at most one maximal path. If there were $\lambda(T) < \Delta$ leaves, we would only have $\binom{\lambda(T)}{2} < \binom{\Delta}{2}$ distinct maximal paths, a contradiction; we must then have $\lambda(T) \geq \Delta$.

Contradiction:

Assume that $\Delta \geq 2$, since the cases of $\Delta = 0$ and $\Delta = 1$ are clearly true. Suppose for the sake of contradiction that there are at most $\lambda(T) < \Delta$ leaves. For each $u_i \in N(v)$, let $p_i$ be a maximum length path beginning with $v, \{v, u_i\}, u_i$. Note that there must be $\Delta$ such paths. We know that any such path $p_i$ must terminate in a leaf $\ell_i$.

By the Pigeonhole Principle, where the pigeons are the terminating leaves of each path and the holes are the $\lambda(T)$ leaves available, we know that, since $\lambda(T) < \Delta$, two paths share the same terminating leaf, say $\ell_\omega$.

This is a contradiction, since there can only exist one unique path between $\ell_\omega$ and $v$.

Induction on the number of vertices:

Let us prove this by induction on the number of vertices in the graph $n$.

We formulate a proposition $P(n)$ which is: in a tree with $n$ vertices and maximum degree $\Delta$, the number of leaves in the tree is at least $\Delta$.

Base Case ($n = 1, 2$ and 3): The case of $n = 1$ is trivial - a graph of just 1 node has maximum degree 0 and at least 0 leaves. There is only one possible tree when $n = 2$: $T = (V, E), V = \{u, v\}, E = \{(u, v)\}$. Here $\Delta = 1$, and we have 2 leaves, so it checks out as required.

There is only one possible tree when $n = 3$: $T = (V, E), V = \{u, v, w\}, E = \{(u, v), (v, w)\}$. Here $\Delta = 2$, and we have 2 leaves, so it checks out as required.
We choose to show three base cases here to avoid a slightly unfortunate edge case in the Induction Step.

**Induction Step:** Assume that (IH) $P(k)$ is true, for some $k \in \mathbb{Z}^+, k \geq 3$. Consider an arbitrary tree $T = (V, E)$ such that $|V| = k + 1$ and it has maximum degree $\Delta$. Let $\ell \in V$ be an arbitrary leaf in $T$ who has some neighbor $a$. Consider $T' = (V', E')$ where $V' = V \setminus \ell$ and $E' = E \setminus \{a, \ell\}$.

We know that $|V'| = k$ and is a tree (since removal of a leaf can never disconnect a tree), so we can apply the Induction Hypothesis on $T'$.

Note that there are two cases here:

1. $a$ was the only vertex of degree $\Delta$ in $T$.
   
   It must be the case then that $a$ has degree $\Delta - 1$ in $T'$ and is of maximum degree. The Induction Hypothesis gives us that $T'$ must have at least $\Delta - 1$ leaves.
   
   Further note if $a$ is a leaf in $T'$, then it must be the case that $n = 3$ (convince yourself of this), and that is already shown to be true by the base case. Hence, going forward we will operate under the assumption that $a$ is not a leaf.
   
   Adding $\ell$ back to $T'$ to reconstruct $T$ increases the number of leaves by one (since $a$ is not a leaf), so we have that $T$ has at least $\Delta$ leaves.

2. There is some vertex in $T'$ that has degree $\Delta$.
   
   By the Induction Hypothesis, we have that $T'$ must have $\Delta$ leaves.
   
   There are two more cases here:
   
   (a) $a$ is a leaf in $T'$
   
   In this case, the addition of $\ell$ does not change the number of leaves, which means we have at least $\Delta$ leaves in $T$, as desired.
   
   (b) $a$ is not a leaf in $T'$
   
   In this case, the addition of $\ell$ increases the number of leaves by 1, which means we have at least $\Delta + 1$ leaves in $T$, which proves our claim.

**Induction on the number of edges:**

You can do a similar procedure to the induction on the number of vertices in order to perform induction on the number of edges. Note that in this case you would consider the subgraph induced by the vertices other than the leaf.

**Strong Induction on the number of edges:**

Let us prove this by induction on the number of edges in the graph $m$.

We formulate a proposition $P(m)$ which is: in a tree with $m$ edges and maximum degree $\Delta$, the number of leaves in the tree is at least $\Delta$.

**Base Case ($m=0, 1, 2$):** There is only one possible tree (of one vertex) when $m = 0$. Here $\Delta = 0$, and we have at least 0 leaves.

There is only one possible tree when $m = 1$: $T = (V, E), V = \{u, v\}, E = \\{\{u, v\}\}$. Here $\Delta = 1$, and we have 2 leaves, so it checks out as required.
There is only one possible tree when \( m = 2 \): \( T = (V, E), V = \{u, v, w\}, E = \{\{u, v\}, \{v, w\}\} \). Here \( \Delta = 2 \), and we have 2 leaves, so it checks out as required.

We choose to show two base cases here to avoid a slightly unfortunate edge case in the Induction Step.

**Induction Step:** Assume that (III) \( P(j) \) is true, for all \( j \in \mathbb{Z}, 1 \leq j \leq k \), for some \( k \in \mathbb{Z}^+ \), \( k \geq 2 \). Let \( T \) be a tree with \( k + 1 \) edges and with a maximum degree \( \Delta \). Let \( v \) be a vertex with degree \( \Delta \), and \( u \) be an arbitrary neighbor of \( v \). Let us consider \( G' = (V', E'), \) where \( V' = V, E' = E \setminus \{\{u, v\}\} \). Note that \( G' \) must have had two connected components \( C_1 \) and \( C_2 \), which are both trees when a subgraph is induced on each of them. Let \( C_1 \) be the component with \( v \), and let \( C_2 \) be the component with \( u \).

There are two cases here:

1. **There is another vertex in \( C_1 \) that has degree \( \Delta \)**
   - From the induction hypothesis, we have that there must be \( \Delta \) leaves in \( C_1 \). Let us reconstruct \( T \) from \( G' \).
   
   There are two cases here:
   - (a) \( |C_2| = 1 \)
     - In this case, if \( v \) is a leaf in \( G' \), then the addition of \( \{u, v\} \) will not change the number of leaves. Therefore we have that \( T \) must have at least \( \Delta \) leaves. If \( v \) is not a leaf, then the addition of \( \{u, v\} \) will add an additional leaf, so we have that \( T \) must have at least \( \Delta + 1 \) leaves.
   - (b) \( |C_2| \geq 2 \)
     - In this case, \( C_2 \) must have two leaves. Hence there are at least \( \Delta + 2 \) leaves in \( G' \). Notice that the addition of the edge \( \{u, v\} \) can decrease the number of leaves by up to 2 (if \( u \) and \( v \) were both leaves in \( G' \)). Hence we have that \( T \) has at least \( \Delta \) leaves, as required.

2. **\( v \) is the only vertex with degree \( \Delta \) in \( T \)**
   - Hence, \( \Delta(C_1) = \Delta - 1 \). From the induction hypothesis, we know that \( C_1 \) must have \( \Delta - 1 \) leaves. We further note that if \( v \) is a leaf in \( G' \), it must be that \( m = 2 \) (convince yourself of this), and we have already shown the validity of this in the base case. We will therefore operate now under the assumption that \( v \) is not a leaf.
   
   There are two cases here:
   - (a) \( |C_2| = 1 \)
     - Since \( v \) is not a leaf, then the addition of \( \{u, v\} \) will add an additional leaf, so we have that \( T \) must have at least \( \Delta \) leaves.
   - (b) \( |C_2| \geq 2 \)
     - In this case, \( C_2 \) must have two leaves. Hence there are at least \( \Delta + 1 \) leaves in \( G' \). Notice that the addition of the edge \( \{u, v\} \) can decrease the number of leaves by up to 1 (if \( u \) is a leaf in \( G' \)). Hence we have that \( T \) has at least \( \Delta \) leaves, as required.

**Using inequalities:**
We know that a tree with \( n \) vertices must have \( n - 1 \) edges. Since the sum of the degrees of all the vertices in a graph must be twice the number of edges, we know that the total of all degrees in the tree must be \( 2n - 2 \).

Let us consider the following partitioning of the vertices in \( V \). Let \( A = \{ v \in V \mid \deg(v) = \Delta \} \), \( B = \{ v \in V \mid 1 < \deg(v) < \Delta \} \), and \( C = \{ v \in V \mid \deg(v) = 1 \} \). Note that \( V = A \cup B \cup C \) and \( A \cap B = \emptyset, A \cap C = \emptyset, B \cap C = \emptyset \). Note that \( C \) is the set of leaves.

\[
2n - 2 = \sum_{v \in V} \deg(v)
\]

\[
= \sum_{v \in A} \deg(v) + \sum_{v \in B} \deg(v) + \sum_{v \in C} \deg(v)
\]

\[
= \Delta \cdot |A| + \sum_{v \in B} \deg(v) + |C|
\]

\[
\geq \Delta \cdot |A| + |C| + 2 \cdot |B|
\]

\[
= \Delta \cdot |A| + |C| + 2 \cdot (n - |A| - |C|) \quad \text{(since } n = |A| + |B| + |C|)\]

\[
= (\Delta - 2) \cdot |A| - |C| + 2n
\]

\[
\geq (\Delta - 2) - |C| + 2n \quad \text{(since } |A| \geq 1)\]

Hence we have established that \( 2n - 2 \geq (\Delta - 2) - |C| + 2n \). Further, we have that:

\[
2n - 2 \geq (\Delta - 2) - |C| + 2n
\]

\[
-2 \geq \Delta - 2 - |C|
\]

\[
|C| \geq \Delta
\]

Hence we have that the number of leaves is at least \( \Delta \).

**Minimal Counterexample:**

Consider a minimal counterexample, i.e. a tree \( T \) which violates this property with the minimum possible number of vertices, say \( m \). We know that the case for \( m = 1, 2 \) can be handled easily, so we may assume that \( m \geq 3 \), i.e. the tree has at least 3 vertices. Now pick an arbitrary leaf \( \ell \) and name its only neighbor in the graph \( v \); remove \( \ell \). Consider the resulting graph \( T' \). Note that \( T' \) has exactly \( m - 1 \) vertices. The following cases can occur:

**Case 1:** \( \Delta(T) = \Delta(T') \).

Note that by removing a single leaf, we can never increase the number of leaves in the graph. It follows that \( T' \) has at most as many leaves as \( T \), i.e.

\[
\lambda(T') \leq \lambda(T) < \Delta(T) = \Delta(T')
\]

But this means \( \lambda(T') < \Delta(T') \) and \( T' \) has \( m - 1 \) vertices. This is a contradiction, as we chose \( T \) to be tree with the fewest vertices which violates the claim.

**Case 2:** \( v \) is a leaf in \( T' \) and \( \Delta(T) \neq \Delta(T') \).

The only vertex whose degree can be affected by removing \( \ell \) is \( v \). Then \( v \) must have degree 1 and all other vertices must have degree \( \leq 1 \). The only trees for which this hold have exactly 1 or 2 vertices;
we already know that these cases do not violate the claim. As such, we have a contradiction (it’s impossible for us to end up in this scenario).

Case 3: $v$ is not a leaf in $T'$ and $\Delta(T) \neq \Delta(T')$. If the maximum degree changes by removing this leaf, that means that it must decrease by exactly one (we cannot increase degree by removing edges and only removed one edge). In other words, $\Delta(T') = \Delta(T) - 1$. Note that the number of leaves in $T'$ is $\lambda(T) - 1$, since we removed $\ell$ and $v$ is not a leaf. It follows that

$$\lambda(T') = \lambda(T) - 1 < \Delta(T) - 1 = \Delta(T')$$

Again, $\lambda(T') < \Delta(T')$ and we have a contradiction, since $T$ is not minimal.

In every case we have a contradiction - it must be the case that the set of counterexample is empty, i.e. there are no trees which violate the claim, and we are finished.
Problem 4: Consider an undirected graph \( G = (V, E) \) and a rooted spanning tree \( T \) with root \( u \). Note that a rooted tree is a tree with a special vertex labelled as the “root” of the tree. Consider the layers of \( T \), which can be denoted as \( l_0, l_1, \ldots, l_k \) where \( l_0 = \{ u \} \). Layers are defined as set of vertices that are the same distance from \( u \). Prove that \( G \) is bipartite iff there are no edges in \( G \) that are between vertices that exist in layers enumerated with the same parity (for example, no edge between \( l_2 \) and \( l_6 \) and no edge between \( l_3 \) and \( l_3 \)).

Solution:

(\( \Leftarrow \)) We first prove the backward direction, namely that if \( G \) is a graph where there are no edges between the vertices that exist in the layers enumerated with the same parity, then \( G \) is bipartite. In order to show that \( G \) is bipartite, we will show that it is 2-colorable.

We construct a 2-coloring of the vertices of \( G \) as follows: color all vertices in even-numbered layers \( (l_0, l_2, \ldots) \) red and all vertices in odd-numbered layers \( (l_1, l_3, \ldots) \) blue. We now show that this 2-coloring is valid, thus showing that \( G \) is 2-colorable and bipartite. Consider any edge \( \{s, t\} \in E \). Since no edge in \( G \) exists between vertices in layers enumerated with the same parity, we know that one of \( s \) and \( t \) belongs to an even-numbered layer, while the other belongs to an odd-numbered layer. Then we know that they are not the same color, thus showing our coloring is valid and proving our claim.

(\( \Rightarrow \)): We now prove the forward direction, namely that if \( G \) is a bipartite graph, then there are no edges in \( G \) between vertices that exist in the layers enumerated with the same parity. Consider a proper coloring of \( G \); assume WLOG that \( u \) is red. We claim that the layers \( l_0, l_1, \ldots, l_k \) must alternate between red and blue. Since every vertex in \( l_0 \) (only \( u \)) is red, and every vertex in \( l_1 \) has an edge to its parent in \( l_0 \), all vertices in \( l_1 \) are blue. Similarly, every vertex in \( l_2 \) must be red. We can show that the following pattern continues: Every vertex in \( l_i \) has to have a different color from its parent in \( l_{i-1} \). Try to prove this using induction! So the red vertices occur precisely in the even layers, and the blue vertices occur precisely in the odd layers.

But we know that no edge can have both endpoints of the same color. Hence, it is impossible to have an edge between two layers of the same parity.

Alternate Solution:

We can also use a proposition from lecture that states, “A graph is bipartite iff it does not contain a cycle of odd length” to prove the forward direction.

(\( \Rightarrow \)): Suppose that \( G \) is a bipartite graph. We want to show that there are no edges in \( G \) between vertices that exist in the layers enumerated with the same parity. Since \( G \) is a bipartite graph, we know by the proposition that \( G \) doesn’t contain a cycle of odd length. Now, assume towards a contradiction that there exists an edge in \( G \) between vertices that exist in the layers enumerated with the same parity.

Assume that there is an edge between some vertex called \( x \) in \( l_a \) and some vertex called \( y \) in \( l_b \) where \( a, b \in \mathbb{N} \) and \( a, b \) have the same parity. WLOG, assume that \( a < b \). Since layers are defined as set of vertices that are the same distance from \( u \), we see that in \( T \), the distance between \( u \) and \( x \) is \( a \) and the distance between \( u \) and \( y \) is \( b \). Since \( a \) and \( b \) have the same parity, \( a + b \) must be even (since we either have even + even = even or odd + odd = even).

If a path from \( u \) to \( x \) is a subpath of a path from \( u \) to \( y \) (i.e. \( x \) is on the path from \( u \) to \( y \)), then a
path from $x$ to $y$ together with a direct edge between $x$ and $y$ will form a cycle of odd length (since we have either even - even + 1 = odd or odd - odd + 1 = odd). If not, then a path from $u$ to $x$ concatenated with a path from $u$ to $y$ concatenated with a direct edge between $x$ and $y$ will form an odd cycle (since we have either even + even + 1 = odd or odd + odd + 1 = odd). Either way, we get a contradiction, because we know that $G$ cannot have an odd cycle.
Problem 5:

Consider a connected graph $G = (V, E)$ and an arbitrary partition of $G$’s vertex set $V$ into nonempty sets $S$ and $V \setminus S$. Prove that if there exists only one edge $e$ between the vertices in $S$ and the vertices in $V \setminus S$, then $e$ must be in every spanning tree of $G$.

Solution:

Consider an arbitrary spanning tree of $G$, say $T$. Since $T$ is a tree, we know that it is connected, and thus there is a path between any pair of vertices.

Consider a vertex $x \in S$ and consider another vertex $y \in V \setminus S$. Because $T$ is connected, there must be a path $P$ from $x$ to $y$ in $T$. Let us consider this path.

We define edges that cross the cut between $S$ and $V \setminus S$ to have an endpoint in $S$ and an endpoint in $V \setminus S$. We know that $P$ goes from a vertex in $S$ to a vertex in $V \setminus S$. Therefore, there must exist an edge in the path that crosses the cut between $S$ and $V \setminus S$ (if not, then the path would always stay in either $S$ or $V \setminus S$, which it clearly doesn’t). However, by our assumption we know that the only edge that crosses the cut is $e$. Therefore, our path $P$ contains $e$ and hence, our tree $T$ must contain $e$. 