**CIS 160**

**Recitation Guide - Week 5**

**Topics Covered:** Strong Induction, Pigeonhole Principle, Probability

**Problem 1:**
Consider any five points $P_1, ..., P_5$ in the interior of a square of length 2. Show that at least two of the points must be at a distance of at most $\sqrt{2}$ apart.

**Solution:**
Consider partitioning the square into four smaller squares, all of equal size (draw a line down from the middle of the top to the bottom edge and another from right to left from the middle of the left to the right edge). Now, each of these squares has side length 1, so the diagonal of each square will be of length $\sqrt{2}$, which is the maximum distance between any two points in the square. Let the pigeons be the points, and the squares be the holes. By the Pigeonhole Principle, we have that there exists at least $\left\lceil \frac{5}{4} \right\rceil = 2$ points in one square. Since the maximum distance between any two points within the same square is $\sqrt{2}$, these two points will be at a distance of at most $\sqrt{2}$ apart.
**Problem 2:** Alice and Bob are playing a game in which there are two non-empty bags with an equal number of marbles in them. In this game, the two players take turns removing marbles from one of the bags. In each turn, the player can remove any positive number of marbles as long as they are all from the same bag. The winner of the game is the player that removes the last marble. In Alice and Bob’s current configuration, both bags initially start with the same number of marbles. Prove that one of them can guarantee a win.

**Solution:**

Consider the following strategy: the player who goes second always removes the same number of marbles as the player who went first, but from the other bag. If Bob goes first, Alice can always win by using this strategy.

Define $P(n)$ to be the claim that this strategy always works for bags that start with $n$ marbles each. We prove our strategy works by strong induction.

**Base Case:** $k = 1$. Bob goes first. His only move is to remove one marble from a bag. Alice then removes the last marble from the other bag. Thus the strategy works.

**Induction Hypothesis:** Assume $P(j)$ is true, for $1 \leq j \leq k$, for some $k \in \mathbb{Z}^+$. 

**Induction Step:** We want to show that the claim still holds if each bag has $k + 1$ marbles. So, we start with two bags containing $k + 1$ marbles each. In Bob’s first move, he can remove $m$ number of marbles for $m \in \mathbb{Z}$, $1 \leq m \leq k + 1$.

**Case 1:** $m = k + 1$ (i.e. Bob removes all the marbles from a bag).

In this case, Alice can just take the $k + 1$ marbles in the other bag. Because she took the last marble, she wins.

**Case 2:** $1 \leq m \leq k$:

Thus, after the first move, the bags contain $k + 1 - m$ and $k + 1$ marbles. According to the strategy, Alice removes $m$ marbles from the other bag so that both bags now contain $k + 1 - m$ marbles. We can view the current state of the game as a new game in which both piles contain $k + 1 - m$ marbles. Since $1 \leq k + 1 - m \leq k$, we can apply the induction hypothesis to state that this strategy will always work.
Problem 3:

A standard 52-card deck consists of cards labelled 2 through 10, an Ace, Jack, Queen and King, each with four suits. A hand consists of five cards drawn from the deck. Tashweena is a wannabe magician who is trying to draw specific hands for her new magic show: The Appearing Pigeon. However, Tashweena can’t quite consistently draw a specific hand, but she has learned how to draw any hand uniformly at random from the Gandhi school of magic.

(a) Calculate the probability that she draws a four of a kind successfully. A hand is considered “four of a kind” if it contains all four suits of a specific label.

(b) Calculate the probability that she draws a full house successfully. A hand is considered “full house” if it contains three cards of the same label and two cards of the some other label (i.e. 3 Aces and 2 8s).

Solution:

(a) The sample space, \( \Omega \), is all the possible ways in which 5 cards can be chosen from the 52 card deck. \(|\Omega| = \binom{52}{5}\). Since we are equally likely to pick any hand from the deck, note that the sample space is uniform.

Let \( A \subseteq \Omega \) be the set of outcomes (events) where she draws a 4 of a kind. Since \( \Omega \) is a uniform sample space, \( \Pr[A] = \frac{|A|}{|\Omega|} \).

We can compute \(|A|\) as follows:

Step 1: Pick a label. \( \binom{13}{1} \) ways

Step 2: Pick the 4 cards having the same label. \( \binom{4}{4} \) ways

Step 3: Pick the suit for the 5th card. \( \binom{4}{1} \) ways

Step 4: Pick the label of the 5th card. \( \binom{12}{1} \) ways (We already picked 4 cards of the same label, there are 12 labels left)

By the Multiplication Rule, \(|A| = \binom{13}{1} \times \binom{4}{4} \times \binom{4}{1} \times \binom{12}{1}\). Thus, \( \Pr[A] = \frac{|A|}{|\Omega|} = \frac{\binom{13}{1} \times \binom{4}{4} \times \binom{4}{1} \times \binom{12}{1}}{\binom{52}{5}} \).

(b) Let \( A \subseteq \Omega \) be the set of outcomes (events) where she draws a full house. Since \( \Omega \) is a uniform sample space, \( \Pr[A] = \frac{|A|}{|\Omega|} \).

We can compute \(|A|\) as follows:

Step 1: Pick a label for the three of a kind. \( \binom{13}{1} \) ways

Step 2: Pick the suits in that triple. \( \binom{4}{3} \) ways

Step 3: Pick a label for the pair. \( \binom{12}{1} \) ways

Step 4: Pick the suits for the pair. \( \binom{4}{2} \) ways

By the Multiplication Rule, \(|A| = \binom{13}{1} \times \binom{4}{3} \times \binom{12}{1} \times \binom{4}{2}\). Thus \( \Pr[A] = \frac{|A|}{|\Omega|} = \frac{\binom{13}{1} \times \binom{4}{3} \times \binom{12}{1} \times \binom{4}{2}}{\binom{52}{5}} \).