CIS 160
Recitation Guide - Week 8

Topics Covered: Strong Induction, Probability

Problem 0: Please fill out the following Mid-Semester Feedback Form: bit.ly/cis160-feedback-s19
This form is completely anonymous and will help us improve the course. Please be honest!
**Problem 1:** A car needs 1 unit of length to park while a truck needs 2 units of length. Assume that cars are indistinguishable and so are trucks. How many distinct car/truck parking patterns are possible along an $n$ unit long sidewalk?

**Solution:**

We write the parking patterns as a string of Cs and Ts. Here are two distinct ways in which 3 cars and 2 trucks can be parked along a sidewalk that is 7 units long: CTCCT and TCTCC.

<table>
<thead>
<tr>
<th>length</th>
<th>patterns</th>
<th>#</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>C</td>
<td>1</td>
</tr>
<tr>
<td>2</td>
<td>CC T</td>
<td>2</td>
</tr>
<tr>
<td>3</td>
<td>CT CCC TC</td>
<td>3</td>
</tr>
<tr>
<td>4</td>
<td>CCT TT CTC CCCC TCC</td>
<td>5</td>
</tr>
<tr>
<td>5</td>
<td>CTT CCCT TCT CCTC TTC CTCC CCCCC TCCC</td>
<td>8</td>
</tr>
</tbody>
</table>

We prove by induction that the number of distinct parking patterns along a sidewalk of length $n \geq 1$ is $F_{n+1}$. It’s a special strong induction with an IH only for $k$ and $k - 1$, and hence two base cases.

*(BASE CASE 1)* $n = 1$. Only 1 pattern, C. $F_2 = 1$. ✓

*(BASE CASE 2)* $n = 2$. 2 patterns, CC and T. $F_3 = 2$. ✓

*(INDUCTION STEP)* Let $k$ arbitrary, $k \in \mathbb{Z}, k \geq 2$.

Assume (IH) that the number of patterns for length $k - 1$ is $F_k$ and that the number of patterns for length $k$ is $F_{k+1}$.

Now consider a pattern $p$ for length $k + 1$. Depending on whether this pattern ends with a car or a truck, we have two cases.

**Case 1.** The last vehicle in $p$ is a car, that is, $p = rC$. Then, $r$ has length $k + 1 - 1 = k$. By IH, there are $F_{k+1}$ distinct $r$’s. Therefore, in this case, we have $F_{k+1}$ distinct patterns.

**Case 2.** The last vehicle in $p$ is a truck, that is, $p = sT$. Then, $s$ has length $k + 1 - 2 = k - 1$. By IH, there are $F_k$ distinct $s$’s therefore $F_k$ distinct $p$’s in this case.

Since these two cases are disjoint, by the addition rule, there are $F_{k+1} + F_k = F_{k+2}$ distinct patterns.
Problem 2:
Compute the probability of the event “when we roll \( n \) (distinguishable) fair dice any \( k \) of the dice show the same number while the other \( n - k \) show numbers different from the one shown by the \( k \) dice”. Assume \( n \geq 3 \) and \( \frac{n}{2} < k < n \).

Solution:
As discussed in class, we have a uniform probability space whose outcomes are sequences of length \( n \) of numbers from \([1 .. 6]\). In other words, the \( \Omega \) is given by the cartesian product of \([1 .. 6] \times \cdots \times [1 .. 6] \) (\( n \) times), i.e., \( \Omega = [1 .. 6]^n \). By the Multiplication Rule, there are \( 6 \times \cdots \times 6 = 6^n \) such sequences so each outcome has probability \( \frac{1}{6^n} \).

Let \( E \) be the event where exactly \( k \) of the dice show the same number. We see that we are trying to find \( \Pr[E] \). To compute the desired probability it suffices, by Lecture 12 slide 15, to count the cardinality of \( E \), i.e., the number of sequences (of interest) in which \( k \) positions have the same number from \( t \in [1 .. 6] \) while the other \( n - k \) position show numbers different from \( t \). Such a sequence can be constructed as follows:

Step 1: Choose \( t \in [1 .. 6] \). This can be done in 6 ways.
Step 2: Choose \( k \) of the \( n \) positions in the sequence. This can be done in \( \binom{n}{k} \) ways.
Step 3: Place \( t \) in each of these positions. This can be done in 1 way.
Step 4: For each of the remaining \( n - k \) positions choose a number from \([1 .. 6] \setminus \{t\}\). We see that there are 5 such numbers, and \( n - k \) positions that we have left to fill. Thus, this can be done in \( 5^{n-k} \) ways.

By the Multiplication Rule, the number of sequences of interest is \( 6 \binom{n}{k} 5^{n-k} \). Hence, the probability we are asked for is given by:

\[
\Pr[E] = \frac{|E|}{|\Omega|} = \left( 6 \binom{n}{k} 5^{n-k} \right) \frac{1}{6^n} = \binom{n}{k} \frac{5^{n-k}}{6^{n-1}}
\]

Aside:
A student from recitation brought up the following point. Why doesn’t our method overcount? For example, let us consider the outcome \( T = (1, 1, 2, 2, 3, 3) \in \Omega \) for \( n = 6 \). We could derive \( T \) by first picking \( t = 1 \) and assigning it to the first two dice. Then in step 4 we could generate \((2, 2, 3, 3)\). Similarly our method allows us to pick \( t \neq 1 \). We will show it is impossible to construct our tuple \( T \) again when \( t \neq 1 \). To see this, notice that because \( k > \frac{n}{2} \) any tuple constructed with \( t \) chosen in step 1 will have more than half of the positions will be filled with \( t \). But our tuple \( T \) had 1 in the more than half of its positions! So no matter what we do in steps 2-4 we could have never constructed \( T \) again.
**Problem 3**  Compute the probability of the event “when we roll two identical 6-sided beige dice the numbers add up to an even number.”

**Solution:**

We first observe, by our discussion in lecture, the sample space for this problem is given by:

\[ \Omega = \{ \{x, y\} \mid x, y \in [1..6], x \neq y \} \cup \{x-x \mid x \in [1..6] \} \]

Let \( E \) be the event where the sum of the two rolls results in an even number. Note that we have \( \binom{6}{2} = 15 \) outcomes in which the dice show different numbers; each of these has probability \( \frac{1}{18} \) by our analysis in lecture. Among these outcomes, the numbers add up to an even number if they are both odd, and there are 3 of these, \( \{1, 3\} \), \( \{1, 5\} \), \( \{3, 5\} \), or if they are both even – there also 3 of these: \( \{2, 4\} \), \( \{2, 6\} \), \( \{4, 6\} \). So that’s 6 outcomes of probability \( \frac{1}{18} \) each in which the numbers are different.

We also have 6 more outcomes in which the die show the same number; each of these has probability \( \frac{1}{36} \), again from lecture. In all these outcomes the numbers add up to an even number, hence we have another 6 outcomes of probability \( \frac{1}{36} \) each.

We now calculate the desired probability using the definition of event:

\[
\Pr[E] = \sum_{w \in E} \Pr(w) = \Pr[\{1, 3\}] + \Pr[\{1, 5\}] + \Pr[\{3, 5\}] + \Pr[\{2, 4\}] + \Pr[\{2, 6\}] + \Pr[\{4, 6\}] + \sum_{x \in [1..6]} \Pr[x-x]
\]

\[
= 6 \times \frac{1}{18} + 6 \times \frac{1}{36} = \frac{1}{3} + \frac{1}{6} = \frac{1}{2}
\]

That’s the same answer that we got in the lecture. Since “adding up to even” is an event in which the die color doesn’t matter, we could have provided a solution that assumes the dice are green-purple rather than beige-beige.