Recitation Guide - Week 6

**Topics Covered:** Graphs

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**Problem 1:** Consider the statement: in any graph, there is an even number of vertices with odd degree. While you proved this in lecture already, now prove it using induction.

**Solution:**

We will prove this statement by induction on the number of edges, $m$. Let $P(m)$ be defined as:

In any graph with $m$ edges, there are an even number of vertices of odd degree.

**Base Case:** $P(0)$ holds, because a graph with no edge has only isolated vertices, that is, vertices of degree 0. Hence, there are an even number (0) of vertices with odd degree.

**Induction Step:** Assume $P(k)$ is true, for an arbitrary $k \in \mathbb{N}$. Now, we want to prove $P(k + 1)$ is true.

Let $G$ be a graph with $k + 1$ edges. Remove an arbitrary edge $e = \{u, v\}$ from $G$ (note that it could be any edge), so that we now have a graph $G'$ with $k$ edges. By the Induction Hypothesis, the number of vertices with odd degree in $G'$ is even. Denote the number of vertices with odd degree in $G'$ to be $2a$, where $a \in \mathbb{N}$. Now put back the edge $e$ that we removed earlier. Observe that doing so increases the degree of vertices $u$ and $v$ by one each. We consider the following three cases:

**Case 1:** Both $u$ and $v$ have odd degree in $G'$. Adding $e$ back would make the degree of both $u$ and $v$ even. Hence, the number of vertices with odd degree becomes $2a - 2$.

**Case 2:** Both $u$ and $v$ have even degree in $G'$. Adding $e$ back would make the degree of both $u$ and $v$ odd. Hence, the number of vertices with odd degree becomes $2a + 2$.

**Case 3:** Exactly one of $u$ and $v$ has odd degree in $G'$. WLOG, assume $u$ has an odd degree and $v$ has an even degree in $G'$. Adding $e$ back would result in $u$ with an even degree and $v$ with an odd degree. Hence, the number of vertices with odd degree would stay unchanged ($2a$).

In all cases, the number of odd degree vertices in $G$ is even. Thus, we have shown our claim is true when $m = k + 1$, concluding our Induction Step and completing our proof.